11 AREAS OF PLANE FIGURES

Milwaukee Art Museum, Purchase, Layton Art Collection

Notice how planes are featured in the design of this modern chair. The designer used geometric principles to create an attractive piece of furniture that can be mass-produced from machine-made materials.

Areas of Polygons

Objectives

1. Understand what is meant by the area of a polygon.
2. Understand the area postulates.
3. Know and use the formulas for the areas of rectangles, parallelograms, triangles, rhombuses, trapezoids, and regular polygons.

11-1 *Areas of Rectangles*

In everyday conversation people often refer to the *area* of a rectangle when what they really mean is the area of a rectangular region.

Rectangle

Rectangular region

We will continue this common practice to simplify our discussion. Thus, when we speak of the area of a triangle, we will mean the area of the triangular region that includes the triangle *and* its interior.

In Chapter 1 we accepted postulates that enable us to express the lengths of segments and the measures of angles as positive numbers. Similarly, the areas of figures are positive numbers with properties given by area postulates.

Postulate 17

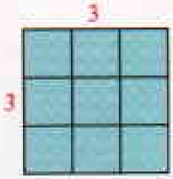
The area of a square is the square of the length of a side. ($A = s^2$)

Length: 1 unit

Area: 1 square unit

By counting,
Area = 9 square units

By using the formula,
Area = $3^2 = 9$ (square units)

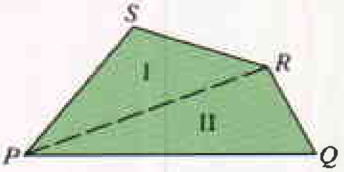


Postulate 18 *Area Congruence Postulate*

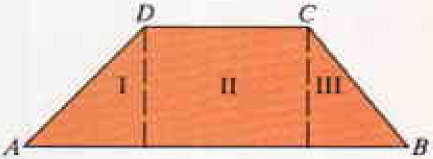
If two figures are congruent, then they have the same area.

Postulate 19 *Area Addition Postulate*

The area of a region is the sum of the areas of its non-overlapping parts.

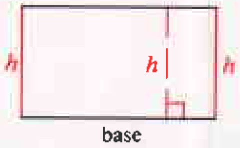


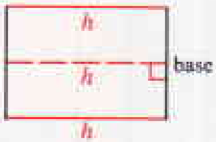
Area of *PQRS* = Area I + Area II

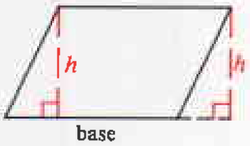


Area of *ABCD* = Area I + Area II + Area III

Any side of a rectangle or other parallelogram can be considered to be a **base.** The length of a base will be denoted by b. In this text the term *base* will be used to refer either to the line segment or to its length. An **altitude** to a base is any segment perpendicular to the line containing the base from any point on the opposite side. The length of an altitude is called the **height** (h). All the altitudes to a particular base have the same length.





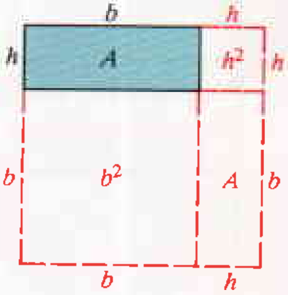


Theorem 11-1

The area of a rectangle equals the product of its base and height. ($A = bh$)

Given: A rectangle with base b and height h

Prove: $A = bh$



Proof:

Building onto the given shaded rectangle, we can draw a large square consisting of these non-overlapping parts:

the given rectangle with area A
a congruent rectangle with area A
a square with area b^2
a square with area h^2

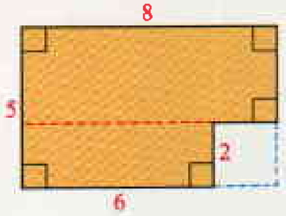
Area of big square $= 2A + b^2 + h^2$	(Area Addition Postulate)
Area of big square $= (b + h)^2 = b^2 + 2bh + h^2$	($A = s^2$)
$2A + b^2 + h^2 = b^2 + 2bh + h^2$	(Substitution Prop.)
$2A = 2bh$	(Subtraction Prop. of =)
$A = bh$	(Division Prop. of =)

Areas are always measured in square units. Some common units of area are the square centimeter (cm^2) and the square meter (m^2). In part (b) of the example below, notice that the unit of length and the unit of area are understood to be "units" and "square units," respectively. It is important to remember that the implied units for length and area are different.

Example Find the area of each figure.

a. A rectangle with base 3.5 cm and height 2 cm

b.



Solution **a.** $A = 3.5(2) = 7\ (cm^2)$

b. *Method 1* (see blue lines) $A = (8 \cdot 5) - (2 \cdot 2) = 40 - 4 = 36$
Method 2 (see red line) $A = (8 \cdot 3) + (6 \cdot 2) = 24 + 12 = 36$

Classroom Exercises

1. Tell what each letter represents in the formula $A = s^2$.
2. Tell what each letter represents in the formula $A = bh$.
3. Find the area and perimeter of a square with sides 5 cm long.
4. The perimeter of a square is 28 cm. What is the area?
5. The area of a square is 64 cm^2. What is the perimeter?

Exercises 6–13 refer to rectangles. Complete the table.

	6.	7.	8.	9.	10.	11.	12.	13.
b	8 cm	4 cm	12 m	?	$3\sqrt{2}$	$4\sqrt{2}$	$5\sqrt{3}$	$x + 3$
h	3 cm	1.2 cm	?	5 cm	2	$\sqrt{2}$	$2\sqrt{3}$	x
A	?	?	36 m^2	55 cm^2	?	?	?	?

14. a. What is the converse of the Area Congruence Postulate?
    b. Is this converse true or false? Explain.
15. a. Draw three noncongruent rectangles, each with perimeter 20 cm. Find the area of each rectangle.
    b. Of all rectangles having perimeter 20 cm, which one do you think has the greatest area? (Give its length and width.)

Written Exercises

Exercises 1–16 refer to rectangles. Complete the tables. p is the perimeter.

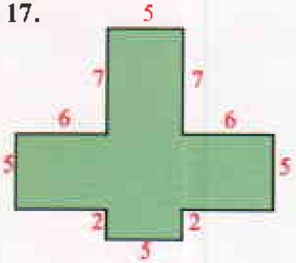
A

	1.	2.	3.	4.	5.	6.	7.	8.
b	12 cm	8.2 cm	16 cm	?	$3\sqrt{2}$	$\sqrt{6}$	$2x$	$4k - 1$
h	5 cm	4 cm	?	8 m	$4\sqrt{2}$	$\sqrt{2}$	$x - 3$	$k + 2$
A	?	?	80 cm^2	120 m^2	?	?	?	?

	9.	10.	11.	12.	13.	14.	15.	16.
b	9 cm	40 cm	16 cm	$x + 5$	$a + 3$	$k + 7$	x	?
h	4 cm	?	?	x	$a - 3$	?	?	y
A	?	?	?	?	?	?	$x^2 - 3x$	$y^2 + 7y$
p	?	100 cm	42 cm	?	?	$4k + 20$	?	?

Consecutive sides of the figures are perpendicular. Find the area of each figure.

B **17.**



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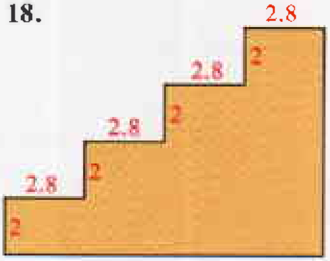

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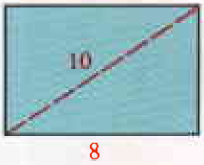

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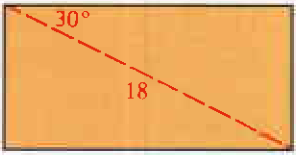

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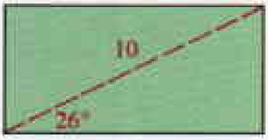
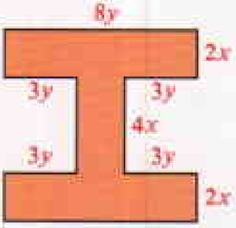


(Give answer correct to the nearest tenth.)

22.

In Exercises 23 and 24 find each area in terms of the variables.

23.



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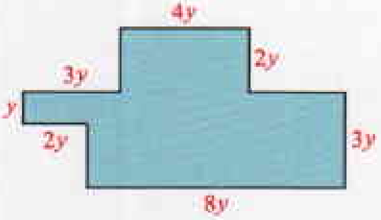


25. Find the area of a square with diagonals of length d.

26. The length of a rectangle is 12 cm more than its width. Find the area of the rectangle if its perimeter is 100 cm.

27. A path 2 m wide surrounds a rectangular garden 20 m long and 12 m wide. Find the area of the path.

28. How much will it cost to blacktop the driveway shown if blacktopping costs \$11.00 per square meter?

Ex. 28

29. A room 28 ft long and 20 ft wide has walls 8 ft high.

a. What is the total wall area?

b. How many gallon cans of paint should be bought to paint the walls if 1 gal of paint covers 300 ft^2?

30. A wooden fence 6 ft high and 220 ft long is to be painted on both sides.

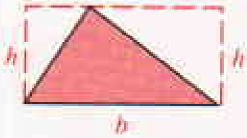
a. What is the total area to be painted?

b. A gallon of a certain type of paint will cover only 200 ft^2 of area for the first coat, but on the second coat a gallon of the same paint will cover 300 ft^2. If the fence is to be given two coats of paint, how many gallons of paint should be bought?

31. A rectangle having area 392 m^2 is twice as long as it is wide. Find its dimensions.

32. The lengths of the sides of three squares are s, $s + 1$, and $s + 2$. If their total area is 365 cm^2, find their total perimeter.

33. Derive a formula for the area of the triangle shown by using the formula for the area of a rectangle.



Ex. 33

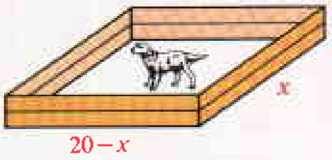
34. The diagonals of a rectangle are 18 cm long and intersect at a 60° angle. Find the area of the rectangle.

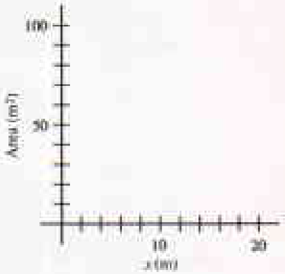
35. a. Suppose you have 40 m of fencing with which to make a rectangular pen for a dog. If one side of the rectangle is x m long, explain why the other side is $(20 - x)$ m long.

b. Express the area of the pen in terms of x.

c. Find the area of the pen for each value of x: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20. Record your answers on a set of axes like the one shown.

d. Give the dimensions of the pen with the greatest area.



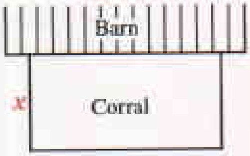


C **36.** A farmer has 100 m of fencing with which to make a rectangular corral. A side of a barn will be used as one side of the corral, as shown in the overhead view.

a. If the width of the corral is x, express the length and the area in terms of x.

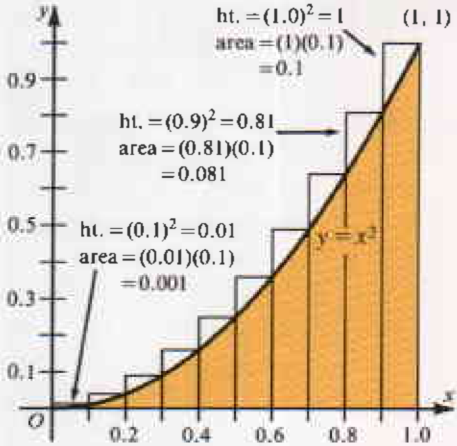
b. Make a graph showing values of x on the horizontal axis and the corresponding areas on the vertical axis.

c. What dimensions give the corral the greatest possible area?



37. Draw a rectangle. Then construct a square with equal area.

♦ Computer Key-In



The shaded region shown is bounded by the graph of $y = x^2$, by the x-axis, and by the vertical line through the points (1, 0) and (1, 1). You can approximate the area of the shaded region by drawing ten rectangles having base vertices at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$, as shown, and computing the sum of the areas of the ten rectangles. The base of each rectangle is 0.1, and the height of each rectangle is given by $y = x^2$.

The following computer program will compute and add the areas of the ten rectangles shown in the diagram. In line 30, Y is the height of each rectangle. In line 40, A gives the current total of all the areas.

```
10  LET X = 0.1
20  FOR N = 1 TO 10
30  LET Y = X ↑ 2
40  LET A = A + Y * 0.1
50  LET X = X + 0.1
60  NEXT N
70  PRINT "AREA IS APPROXIMATELY ";A
80  END
```

If the program is run, the computer will print

```
AREA IS APPROXIMATELY 0.385
```

Exercises

1. A better approximation can be found by using 100 smaller rectangles with base vertices at 0, 0.01, 0.02, 0.03, . . . , 1.00. Change lines 10, 20, 40, and 50 as follows:

```
10  LET X = 0.01
20  FOR N = 1 TO 100
40  LET A = A + Y * 0.01
50  LET X = X + 0.01
```

RUN the program to approximate the area of the shaded region.

2. Modify the given program so that it will use 1000 rectangles with base vertices at 0, 0.001, 0.002, 0.003, . . . , 1.000 to approximate the area of the shaded region. RUN the program.

3. Is the actual area of the shaded region more or less than the value given by the computer program? Explain.

11-2 Areas of Parallelograms, Triangles, and Rhombuses

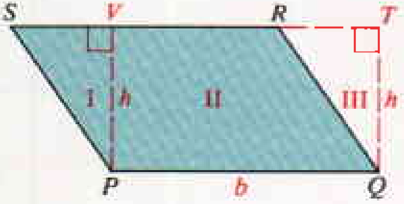
Although proofs of most area formulas are easy to understand, detailed formal proofs are long and time consuming. Therefore, we will show the key steps of each proof.

Theorem 11-2

The area of a parallelogram equals the product of a base and the height to that base. ($A = bh$)

Given: ▱$PQRS$

Prove: $A = bh$



Key steps of proof:

1. Draw altitudes $\overline{PV}$ and $\overline{QT}$, forming two rt. △.
2. Area I = Area III ($\triangle PSV \cong \triangle QRT$ by HL or AAS)
3. Area of ▱$PQRS$ = Area II + Area I
   = Area II + Area III
   = Area of rect. $PQTV$
   = bh

Example 1 Find the area of the parallelogram shown.

Solution Notice the 45°-45°-90° triangle.

$$h = \frac{8}{\sqrt{2}} = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$$

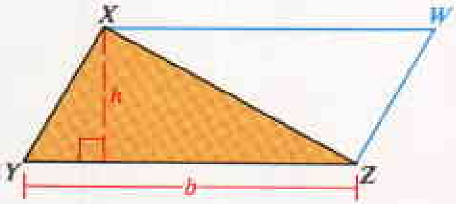
$$A = bh = 10 \cdot 4\sqrt{2} = 40\sqrt{2}$$

Theorem 11-3

The area of a triangle equals half the product of a base and the height to that base. ($A = \frac{1}{2}bh$)

Given: $\triangle XYZ$

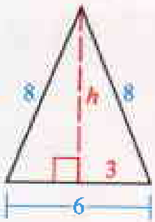
Prove: $A = \frac{1}{2}bh$



Key steps of proof:

1. Draw $\overline{XW} \parallel \overline{YZ}$ and $\overline{ZW} \parallel \overline{YX}$, forming ▱$WXYZ$.
2. $\triangle XYZ \cong \triangle ZWX$ (SAS or SSS)
3. Area of $\triangle XYZ = \frac{1}{2} \cdot$ Area of ▱$WXYZ$
   $= \frac{1}{2}bh$

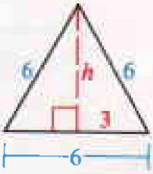
Example 2 Find the area of a triangle with sides 8, 8, and 6.



Solution Draw the altitude to the base shown. Since the triangle is isosceles, this altitude bisects the base.

$h^2 + 3^2 = 8^2$ (Pythagorean Theorem)
$h^2 = 64 - 9 = 55$
$h = \sqrt{55}$
$A = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot \sqrt{55} = 3\sqrt{55}$

Example 3 Find the area of an equilateral triangle with side 6.



Solution Draw an altitude. Two 30°-60°-90° triangles are formed.

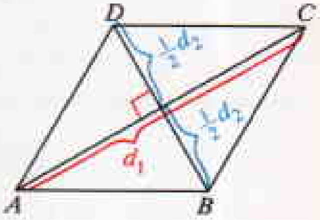
$h = 3\sqrt{3}$
$A = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot 3\sqrt{3} = 9\sqrt{3}$

Theorem 11-4

The area of a rhombus equals half the product of its diagonals. ($A = \frac{1}{2}d_1d_2$)

Given: Rhombus $ABCD$ with diagonals d_1 and d_2

Prove: $A = \frac{1}{2}d_1d_2$



Key steps of proof:

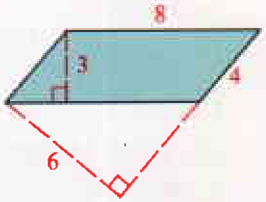
1. $\triangle ADC \cong \triangle ABC$ (SSS)
2. Since $\overline{DB} \perp \overline{AC}$, the area of $\triangle ADC = \frac{1}{2}bh = \frac{1}{2} \cdot d_1 \cdot \frac{1}{2}d_2 = \frac{1}{4}d_1d_2$.
3. Area of rhombus $ABCD = 2 \cdot \frac{1}{4}d_1d_2 = \frac{1}{2}d_1d_2$

Classroom Exercises

1. The area of the parallelogram can be found in two ways:

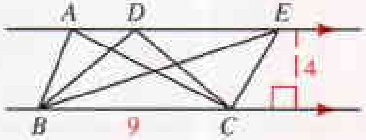
a. $A = 8 \cdot$ __?__ $=$ __?__

b. $A = 4 \cdot$ __?__ $=$ __?__



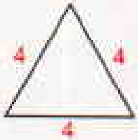
2. Find the areas of $\triangle ABC$, $\triangle DBC$, and $\triangle EBC$.

3. Give two formulas that can be used to find the area of a rhombus. (*Hint:* Every rhombus is also a __?__.)



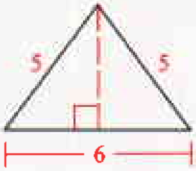
Find the area of each figure.

4.



5.

6.



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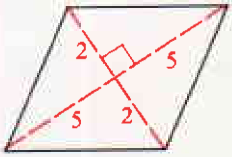

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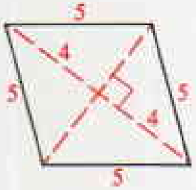

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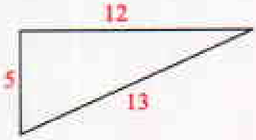


Written Exercises

In Exercises 1–20 find the area of each figure.

A **1.** A triangle with base 5.2 m and corresponding height 11.5 m

2. A triangle with sides 3, 4, and 5

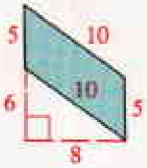
3. A parallelogram with base $3\sqrt{2}$ and corresponding height $2\sqrt{2}$

4. A rhombus with diagonals 4 and 6

5. An equilateral triangle with sides 8 ft

6. An isosceles triangle with sides 10, 10, and 16

7.



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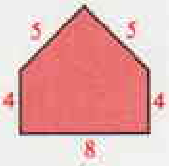

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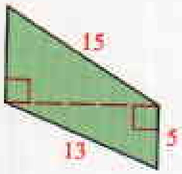


10. An isosceles triangle with base 10 and perimeter 36

11. An isosceles right triangle with hypotenuse 8

12. An equilateral triangle with perimeter 18

B **13.** A parallelogram with a 45° angle and sides 6 and 10

14. A rhombus with a 120° angle and sides 6 cm

15. A 30°-60°-90° triangle with hypotenuse 10

16. An equilateral triangle with height 9

17. A rhombus with perimeter 68 and one diagonal 30

18. A regular hexagon with perimeter 60

19. A square inscribed in a circle with radius r

20. A rectangle with length 16 inscribed in a circle with radius 10

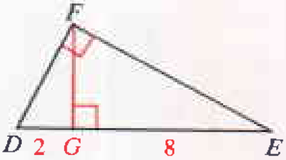
In Exercises 21–24 use a calculator or the trigonometry table on page 311 to find the area of each figure to the nearest tenth.

21.

22.

23.

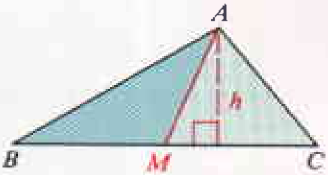
24. An isosceles triangle with a 32° vertex angle and a base of 8 cm



25. $\overline{FG}$ is the altitude to the hypotenuse of $\triangle DEF$. Name three similar triangles and find their areas. (*Hint*: See Theorem 8-1 and Corollary 1 on pages 285–286.)

26. If the area of $\square PQRS$ is 36 and T is a point on $\overline{PQ}$, find the area of $\triangle RST$. (*Hint*: Draw a diagram.)

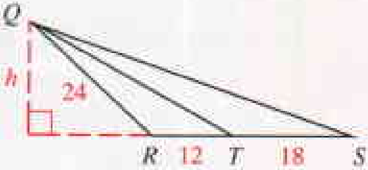
In Exercises 27 and 28, $\overline{AM}$ is a median of $\triangle ABC$.



27. If $BC = 16$ and $h = 5$, find the areas of $\triangle ABC$ and $\triangle AMB$.

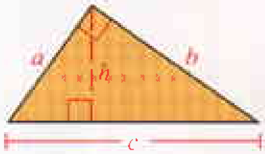
28. Prove: Area of $\triangle AMB = \frac{1}{2} \cdot$ Area of $\triangle ABC$

29. **a.** Find the ratio of the areas of $\triangle QRT$ and $\triangle QTS$.
   b. If the area of $\triangle QRS$ is 240, find the length of the altitude from S to $\overleftrightarrow{QR}$.

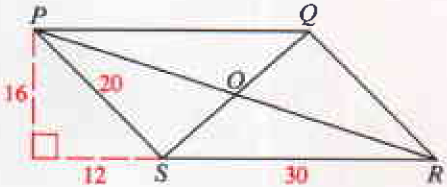


30. An isosceles triangle has sides that are 5 cm, 5 cm, and 8 cm long. Find its area and the lengths of the three altitudes.

31. **a.** Find the area of the right triangle in terms of a and b.
   b. Find the area of the right triangle in terms of c and h.
   c. Solve for h in terms of the other variables.
   d. A right triangle has legs 6 and 8. Find the lengths of the altitude and the median to the hypotenuse.

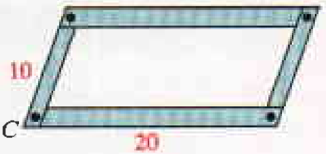


32. Use the diagram at the right.
   a. Find the area of $\square PQRS$.
   b. Find the area of $\triangle PSR$.
   c. Find the area of $\triangle OSR$. (*Hint*: Refer to $\triangle PSR$ and use Exercise 28.)
   d. What is the area of $\triangle PSO$?
   e. What must the area of $\triangle POQ$ be? Why? What must the area of $\triangle OQR$ be?
   f. State what you have shown in parts (a)–(e) about how the diagonals of a parallelogram divide the parallelogram.



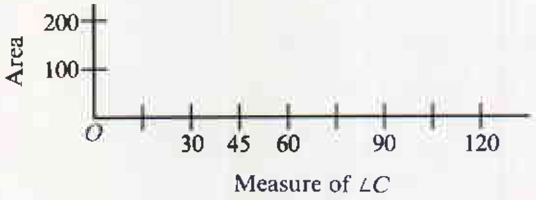
33. **a.** An equilateral triangle has sides of length s. Show that its area is $\frac{s^2}{4}\sqrt{3}$.
   b. Find the area of an equilateral triangle with side 7.

34. Think of a parallelogram made with cardboard strips and hinged at each vertex so that the measure of $\angle C$ will vary. Find the area of the parallelogram for each measure of $\angle C$ given in parts (a)–(e).



a. 30 **b.** 45 **c.** 60 **d.** 90 **e.** 120

f. Approximate your answers to parts (b), (c), and (e) by using $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7$. Then record your answers to parts (a)–(e) on a set of axes like the one below.

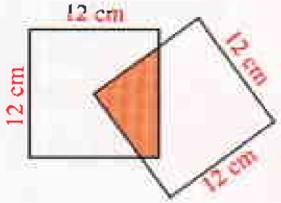


35. The area of a rhombus is 100. Find the length of the two diagonals if one is twice as long as the other.

36. The base of a triangle is 1 cm longer than its altitude. If the area of the triangle is 210 cm^2, how long is the altitude?

C **37.** Find the area of quadrilateral *ABCD* given $A(2, -2)$, $B(6, 4)$, $C(-1, 5)$, and $D(-5, 2)$.

38. Two squares each with sides 12 cm are placed so that a vertex of one lies at the center of the other. Find the area of the shaded region.



39. The diagonals of a parallelogram are 82 cm and 30 cm. One altitude is 18 cm long. Find the two possible values for the area.

For Exercises 40–42, draw a scalene triangle *ABC*.

40. Construct an isosceles triangle whose area is equal to the area of $\triangle ABC$.

41. Construct an isosceles right triangle whose area is equal to the area of $\triangle ABC$.

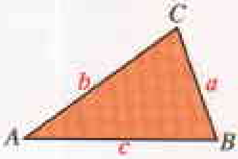
42. Construct an equilateral triangle whose area is equal to the area of $\triangle ABC$.

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

Draw any quadrilateral and connect the midpoints of its sides. You should get a parallelogram (see Exercise 11, page 186). Compare the area of the original quadrilateral and the area of this parallelogram. What do you notice? Can you explain why this is true?

◆ Calculator Key-In



More than 2000 years ago, Heron, a mathematician from Alexandria, Egypt, derived a formula for finding the area of a triangle when the lengths of its sides are known. This formula is known as **Heron's Formula.** To find the area of $\triangle ABC$ using this formula:

Step 1 Find the *semiperimeter* $s = \frac{1}{2}(a + b + c)$.

Step 2 Area $= A = \sqrt{s(s - a)(s - b)(s - c)}$

Example If $a = 5$, $b = 6$, and $c = 7$, find the area of $\triangle ABC$.

Solution *Step 1* $s = \frac{1}{2}(5 + 6 + 7) = 9$

Step 2
$$\begin{aligned} A &= \sqrt{s(s - a)(s - b)(s - c)} \\ &= \sqrt{9(9 - 5)(9 - 6)(9 - 7)} \\ &= \sqrt{9 \cdot 4 \cdot 3 \cdot 2} \\ &= 6\sqrt{6} \end{aligned}$$

It is convenient to use a calculator when evaluating areas by using Heron's Formula. A calculator gives 14.7 as the approximate area of the triangle in the example above.

Exercises

The lengths of the sides of a triangle are given. Use a calculator to find the area and the three heights of the triangle, each correct to three significant digits. (*Hint*: $h = \frac{2A}{b}$.)

1. 9, 10, 11
2. 5, 7, 8
3. 6, 11, 13
4. 15, 16, 17
5. 6.3, 7.2, 10.1
6. 68, 77, 105
7. 5.5, 6.5, 10
8. 12, 18, 27

Use two different methods to find the exact area of each triangle whose sides are given.

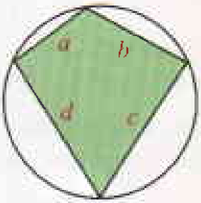
9. 3, 4, 5
10. 6, 6, 6
11. 13, 13, 10
12. 29, 29, 42

13. Something strange happens when Heron's Formula is used with $a = 47$, $b = 38$, and $c = 85$. Why does this occur?

14. Heron also derived the following formula for the area of an inscribed quadrilateral with sides a, b, c, and d:

$A = \sqrt{(s - a)(s - b)(s - c)(s - d)}$,
where the semiperimeter $s = \frac{1}{2}(a + b + c + d)$

Use this formula to find the area of an isosceles trapezoid with sides 10, 10, 10, and 20 that is inscribed in a circle.



11-3 *Areas of Trapezoids*

An **altitude** of a trapezoid is any segment perpendicular to a line containing a base from a point on the opposite base. Since the bases are parallel, all altitudes have the same length, called the *height* (h) of the trapezoid.

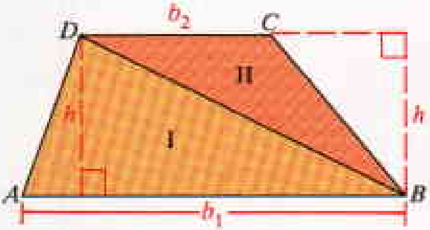
Theorem 11-5

The area of a trapezoid equals half the product of the height and the sum of the bases. ($A = \frac{1}{2}h(b_1 + b_2)$)

Key steps of proof:

1. Draw diagonal $\overline{BD}$ of trap. $ABCD$, forming two triangular regions, I and II, each with height h.
2. Area of trapezoid = Area I + Area II

$$\begin{aligned}\text{Area of trapezoid} &= \text{Area I} + \text{Area II}\\ &= \tfrac{1}{2}b_1h + \tfrac{1}{2}b_2h\\ &= \tfrac{1}{2}h(b_1 + b_2)\end{aligned}$$



Example 1 Find the area of a trapezoid with height 7 and bases 12 and 8.

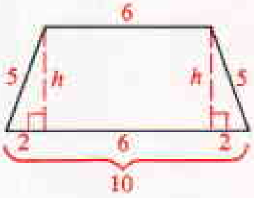
Solution $A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2} \cdot 7 \cdot (12 + 8) = 70$

Example 2 Find the area of an isosceles trapezoid with legs 5 and bases 6 and 10.

Solution When you draw the two altitudes shown, you get a rectangle and two congruent right triangles. The segments of the lower base must have lengths 2, 6, and 2. First find h:

$$\begin{aligned}h^2 + 2^2 &= 5^2\\ h^2 &= 21\\ h &= \sqrt{21}\end{aligned}$$

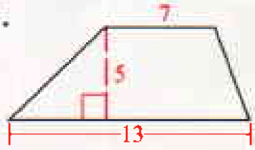
Then find the area: $A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}\sqrt{21}(10 + 6) = 8\sqrt{21}$



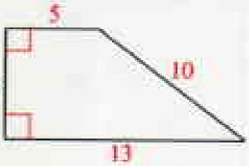
Classroom Exercises

Find the area of each trapezoid and the length of the median.

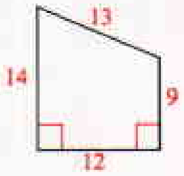
1.



2.



3.



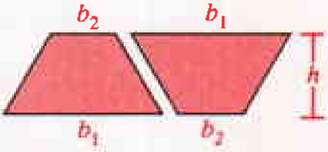
4. Use your answers from Exercises 1–3 to explain why the area of a trapezoid can be given by the formula

$$\textit{Area} = \textit{height} \times \textit{median}.$$

5. Does the median of a trapezoid divide it into two regions of equal area?

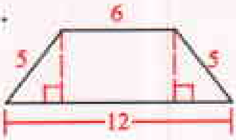
6. Does the segment joining the midpoints of the parallel sides of a trapezoid divide it into two regions of equal area?

7. **a.** If the congruent trapezoids shown are slid together, what special quadrilateral is formed?
   b. Use your answer in part (a) to derive the formula $A = \frac{1}{2}h(b_1 + b_2)$.



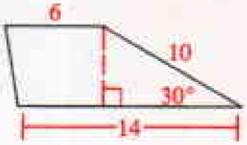
Find the area of each trapezoid.

8.



9.

10.



Written Exercises

Exercises 1–8 refer to trapezoids and *m* is the length of the median. Complete the table.

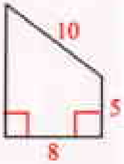
A

	1.	2.	3.	4.	5.	6.	7.	8.
b_1	12	6.8	$3\frac{1}{6}$	45	27	3	7	?
b_2	8	3.2	$4\frac{1}{3}$	15	9	?	?	$3k$
h	7	6.1	$1\frac{3}{5}$	?	?	3	$9\sqrt{2}$	$5k$
A	?	?	?	300	90	12	$36\sqrt{2}$	$45k^2$
m	?	?	?	?	?	?	?	?

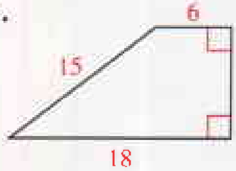
9. A trapezoid has area 54 and height 6. How long is its median?

In Exercises 10–18, find the area of each trapezoid.

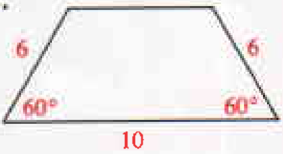
10.



11.



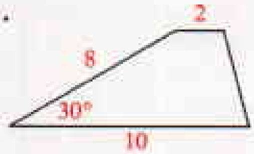
12.



13.

14.

15.



16. An isosceles trapezoid with legs 13 and bases 10 and 20

17. An isosceles trapezoid with legs 10 and bases 10 and 22

18. A trapezoid with bases 8 and 18 and 45° base angles

Use a calculator or the trigonometry table on page 311 to find the area of each trapezoid to the nearest tenth.

B **19.**

20.

21.

22. The legs of an isosceles trapezoid are 10 cm. The bases are 9 cm and 21 cm. Find the area of the trapezoid and the lengths of the diagonals.

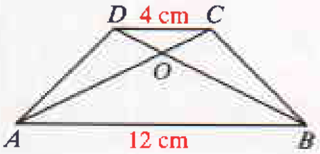
23. An isosceles trapezoid has bases 12 and 28. The area is 300. Find the height and the perimeter.

24. *ABCD* is a trapezoid with bases 4 cm and 12 cm, as shown. Find the ratio of the areas of:

a. $\triangle ABD$ and $\triangle ABC$

b. $\triangle AOD$ and $\triangle BOC$

c. $\triangle ABD$ and $\triangle ADC$



25. *ABCDEF* is a regular hexagon with side 12. Find the areas of the three regions formed when diagonals $\overline{AC}$ and $\overline{AD}$ are drawn.

26. An isosceles trapezoid with bases 12 and 16 is inscribed in a circle of radius 10. The center of the circle lies in the interior of the trapezoid. Find the area of the trapezoid.

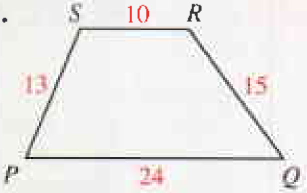
27. A trapezoid of area 100 cm^2 has bases of 5 cm and 15 cm. Find the areas of the two triangles formed by extending the legs until they intersect.

C **28.** Draw a non-isosceles trapezoid. Then construct an isosceles trapezoid with equal area.

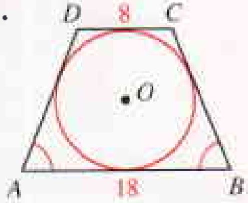
Find the exact area of each trapezoid. In Exercise 31, $\odot O$ is inscribed in quadrilateral *ABCD*.

29.

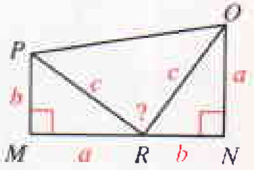
30.



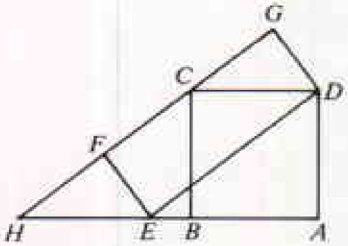
31.



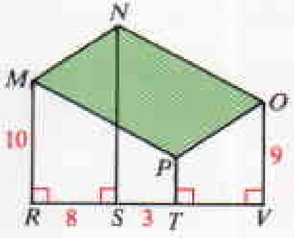
32. President James Garfield discovered a proof of the Pythagorean Theorem in 1876 that used a diagram like the one at the right. Refer to the diagram and write your own proof of the Pythagorean Theorem. (*Hint*: Express the area of quad. *MNOP* in two ways.)



33. Show that the area of square *ABCD* equals the area of rectangle *EFGD*.



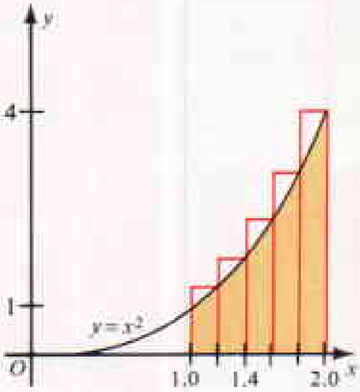
★ **34.** If *NS* = 16, find the area of ▱*MNOP*.

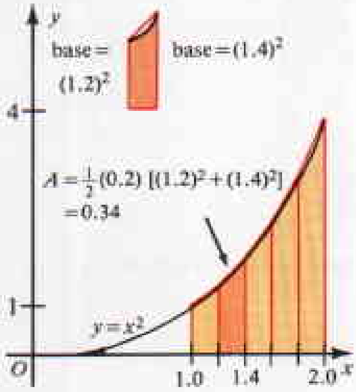


◆ Computer Key-In

The shaded region shown below is bounded by the graph of $y = x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$. The area of this region can be approximated by drawing rectangles. (See the Computer Key-In, page 428.) This area can also be approximated by drawing trapezoids. The curve $y = x^2$ has been exaggerated slightly to better show the trapezoids. The diagrams below suggest that you can obtain a closer approximation for the area by using trapezoids than by using rectangles.

Let us approximate the area using five rectangles and five trapezoids. The base of each rectangle is 0.2 and the height is given by $y = x^2$.





For each trapezoid in the diagram at the right above, the parallel bases are vertical segments from the x-axis to the curve $y = x^2$. The altitude is a horizontal segment with length 0.2. For example, in the second trapezoid, the bases are $(1.2)^2$ and $(1.4)^2$, respectively, and the height is 0.2.

The area of the shaded region is first approximated by the sum of the areas of the five rectangles and then by the sum of the areas of the five trapezoids. Compare these approximations. (Calculus can be used to prove that the exact area is $\frac{7}{3}$. Note that $\frac{7}{3} \approx 2.33$.)

Area approximated by five rectangles:
$A \approx (1.2)^2(0.2) + (1.4)^2(0.2) + (1.6)^2(0.2) + (1.8)^2(0.2) + (2.0)^2(0.2)$
$A \approx 2.64$

Area approximated by five trapezoids:

$$A \approx \tfrac{1}{2}(0.2)[(1.0)^2 + (1.2)^2] + \tfrac{1}{2}(0.2)[(1.2)^2 + (1.4)^2] + \tfrac{1}{2}(0.2)[(1.4)^2 + (1.6)^2] + \tfrac{1}{2}(0.2)[(1.6)^2 + (1.8)^2] + \tfrac{1}{2}(0.2)[(1.8)^2 + (2.0)^2]$$

$A \approx 2.34$

The following computer program will compute and add the areas of the five trapezoids shown in the diagram on the preceding page.

```
10  LET X = 1
20  FOR N = 1 TO 5
30  LET B1 = X ↑ 2
40  LET B2 = (X + 0.2) ↑ 2
50  LET A = A + 0.5 * 0.2 * (B1 + B2)
60  LET X = X + 0.2
70  NEXT N
80  PRINT "AREA IS APPROXIMATELY ";A
90  END
```

Exercises

1. A better approximation can be found by using 100 smaller trapezoids with base vertices at 1.00, 1.01, 1.02, . . . , 1.99, 2.00. Change lines 20, 40, 50, and 60 as follows:

```
20  FOR N = 1 TO 100
40  LET B2 = (X + 0.01) ↑ 2
50  LET A = A + 0.5 * 0.01 * (B1 + B2)
60  LET X = X + 0.01
```

RUN the program to approximate the area of the shaded region.

2. Modify the given computer program so that it will use 1000 trapezoids with base vertices at 1.000, 1.001, 1.002, . . . , 2.000 to approximate the area of the shaded region. RUN the program.

3. Modify the given computer program so that it will use ten trapezoids to approximate the area of the region that is bounded by the graph of $y = x^2$, the x-axis, and *the vertical lines* $x = 0$ *and* $x = 1$. RUN the program. Compare your answer with that obtained on page 428, where ten rectangles were used. (*Note*: Calculus can be used to prove that the exact area is $\frac{1}{3}$.)

Mixed Review Exercises

Complete.

1. In $\odot O$, if the measure of central angle AOB is 52, then the measure of arc AB is $\underline{\quad ? \quad}$.

2. In $\odot P$, if the measure of inscribed angle RST is 73, then the measure of arc RT is $\underline{\quad ? \quad}$.

3. The measure of each interior angle of a regular octagon is $\underline{\quad ? \quad}$.

4. If the measure of each exterior angle of a regular polygon is 20, then the polygon has $\underline{\quad ? \quad}$ sides.

5. In a 45°-45°-90° triangle with legs 20 cm long, the length of the altitude to the hypotenuse is $\underline{\quad ? \quad}$.

6. In a 30°-60°-90° triangle with hypotenuse 30 cm long, the lengths of the legs are $\underline{\quad ? \quad}$ and $\underline{\quad ? \quad}$.

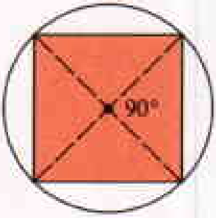
7. In an isosceles triangle with vertex angle of 60° and legs 10 m long, the length of the base is $\underline{\quad ? \quad}$.

8. In $\triangle ABC$ if $m\angle C = 90$, $AC = 8$, and $AB = 17$, then $\cos B = \underline{\quad ? \quad}$.

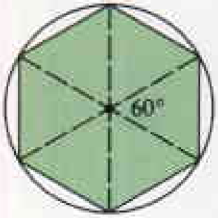
11-4 *Areas of Regular Polygons*

The beautifully symmetrical designs of kaleidoscopes are produced by mirrors that reflect light through loose particles of colored glass. Since the body of a kaleidoscope is a tube, the designs always appear to be inscribed in a circle. The photograph of a kaleidoscope pattern at the right suggests a regular hexagon.

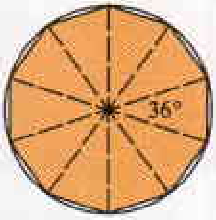
Given any circle, you can inscribe in it a regular polygon of any number of sides. The diagrams below show how this can be done.



Square in circle: draw four 90° central angles.



Regular hexagon in circle: draw six 60° central angles.



Regular decagon in circle: draw ten 36° central angles.

It is also true that if you are given any regular polygon, you can circumscribe a circle about it. This relationship between circles and regular polygons leads to the following definitions:

The **center of a regular polygon** is the center of the circumscribed circle.

The **radius of a regular polygon** is the distance from the center to a vertex.

A **central angle of a regular polygon** is an angle formed by two radii drawn to consecutive vertices.

The **apothem of a regular polygon** is the (perpendicular) distance from the center of the polygon to a side.

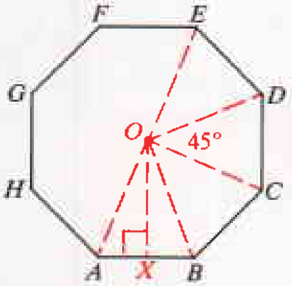
Center of regular octagon: O

Radius: OA, OB, OC, and so on

Central angle: $\angle AOB$, $\angle BOC$, and so on

Measure of central angle: $\frac{360}{8} = 45$

Apothem: OX



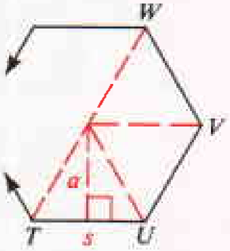
If you know the apothem and the perimeter of a regular polygon, you can use the next theorem to find the area of the polygon.

Theorem 11-6

The area of a regular polygon is equal to half the product of the apothem and the perimeter. ($A = \frac{1}{2}ap$)

Given: Regular n-gon $TUVW$ . . . ; apothem a; side s;
perimeter p; area A

Prove: $A = \frac{1}{2}ap$



Key steps of proof:

1. If all radii are drawn, n congruent triangles are formed.
2. Area of each $\triangle = \frac{1}{2}sa$
3. $A = n(\frac{1}{2}sa) = \frac{1}{2}a(ns)$
4. Since $ns = p$, $A = \frac{1}{2}ap$.

Example 1 Find the area of a regular hexagon with apothem 9.

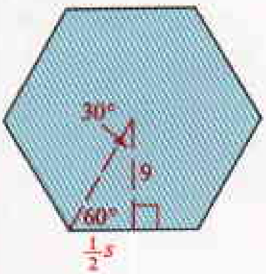
Solution Use 30°-60°-90° $\triangle$ relationships.

$$\frac{1}{2}s = \frac{9}{\sqrt{3}} = 3\sqrt{3}$$

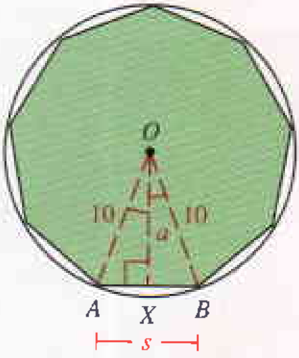
$$s = 6\sqrt{3}; \quad p = 36\sqrt{3}$$

$$A = \frac{1}{2}ap = \frac{1}{2} \cdot 9 \cdot 36\sqrt{3}$$

$$= 162\sqrt{3}$$



Example 2 Find the area of a regular polygon with 9 sides inscribed in a circle with radius 10.



Solution $m\angle AOB = \frac{360}{9} = 40;\ m\angle AOX = 20$

Use trigonometry to find a and s:

$$\cos 20^\circ = \frac{a}{10} \qquad \sin 20^\circ = \frac{\frac{1}{2}s}{10}$$

$$a = 10 \cdot \cos 20^\circ \qquad \tfrac{1}{2}s = 10 \sin 20^\circ$$

$$a \approx 10(0.9397) \qquad s \approx 20(0.3420)$$

$$a \approx 9.397 \qquad s \approx 6.840$$

To find the area of the polygon, use either of two methods:

Method 1 Area of polygon $= 9 \cdot \text{area of } \triangle AOB$

$$= 9 \cdot \tfrac{1}{2}sa$$

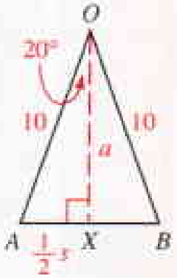
$$\approx \tfrac{9}{2}(6.840)(9.397)$$

$$\approx 289$$

Method 2 Area of polygon $= \frac{1}{2}ap$

$$\approx \tfrac{1}{2}(9.397)(9 \cdot 6.840)$$

$$\approx 289$$



Classroom Exercises

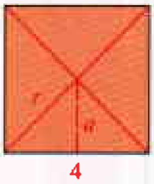
1. Find the measure of a central angle of a regular polygon with **(a)** 10 sides, **(b)** 15 sides, **(c)** 360 sides, and **(d)** n sides.

Find the perimeter and the area of each regular polygon described.

2. A regular octagon with side 4 and apothem a
3. A regular pentagon with side s and apothem 3
4. A regular decagon with side s and apothem a
5. Explain why the apothem of a regular polygon must be less than the radius.

For each regular polygon shown, find (a) the perimeter, (b) the measure of a central angle, (c) the apothem a, (d) the radius r, and (e) the area A.

6.



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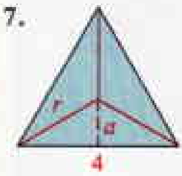
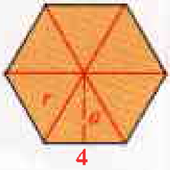


8.



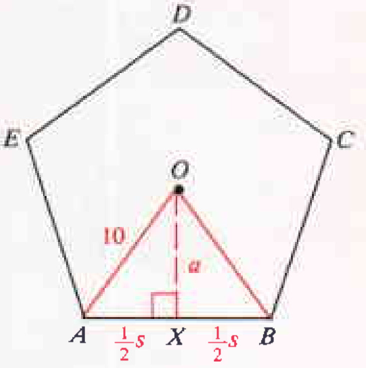
9. *ABCDE* is a regular pentagon with radius 10.
  a. Find the measure of $\angle AOB$.
  b. Explain why $m\angle AOX = 36$.

Note: For parts (c)–(e), use a calculator or the table on page 311.

c. $\cos 36° = \frac{a}{?}$. To the nearest tenth, $a \approx$ _?_.

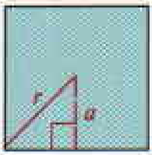
d. $\sin 36° = \frac{\frac{1}{2}s}{?}$. To the nearest tenth, $s \approx$ _?_.

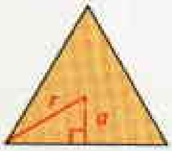
e. Find the perimeter and area of the pentagon.

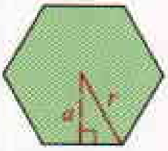


Written Exercises

Copy and complete the tables for the regular polygons shown. In these tables, *p* represents the perimeter and *A* represents the area.







	r	*a*	*A*
A **1.**	$8\sqrt{2}$	?	?
2.	?	5	?
3.	?	?	49
4.	?	$\sqrt{6}$	?

	r	*a*	*p*	*A*
5.	6	?	?	?
6.	?	4	?	?
7.	?	?	12	?
8.	?	?	$9\sqrt{3}$	?

	r	*a*	*p*	*A*
9.	4	?	?	?
10.	?	$5\sqrt{3}$	?	?
11.	?	6	?	?
12.	?	?	$12\sqrt{3}$	?

Find the area of each polygon.

B **13.** An equilateral triangle with radius $4\sqrt{3}$

14. A square with radius $8k$

15. A regular hexagon with perimeter 72

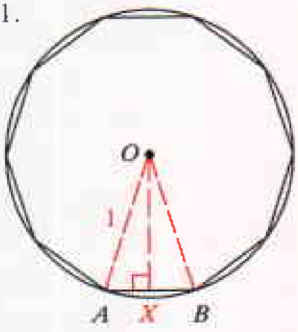
16. A regular hexagon with apothem 4

17. A regular decagon is shown inscribed in a circle with radius 1.
  a. Explain why $m\angle AOX = 18$.
  b. Use a calculator or the table on page 311 to evaluate *OX* and *AX* below.

$\sin 18° = \frac{AX}{1}$, so $AX \approx$ _?_.

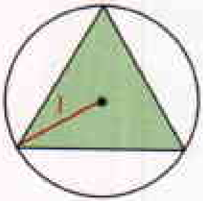
$\cos 18° = \frac{?}{?}$, so $OX \approx$ _?_.

  c. Perimeter of decagon $\approx$ _?_
  d. Area of $\triangle AOB \approx$ _?_
  e. Area of decagon $\approx$ _?_

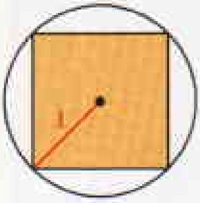


Three regular polygons are inscribed in circles with radii 1. Find the apothem, the perimeter, and the area of each polygon. Use $\sqrt{3} \approx 1.732$ and $\sqrt{2} \approx 1.414$.

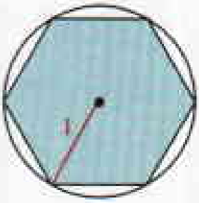
18.



19.

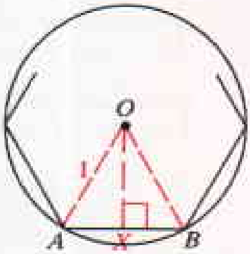


20.



21. Find the perimeter and area of a regular dodecagon (12 sides) inscribed in a circle with radius 1. Use the procedure suggested by Exercise 17.

C **22.** A regular polygon with n sides is inscribed in a circle with radius 1.



a. Explain why $m\angle AOX = \frac{180}{n}$.

b. Show that $AX = \sin\left(\frac{180}{n}\right)^{\circ}$.

c. Show that $OX = \cos\left(\frac{180}{n}\right)^{\circ}$.

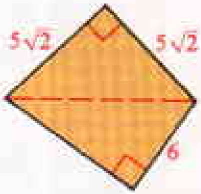
d. Show that the perimeter of the polygon is $p = 2n \cdot \sin\left(\frac{180}{n}\right)^{\circ}$.

e. Show that the area of the polygon is $A = n \cdot \sin\left(\frac{180}{n}\right)^{\circ} \cdot \cos\left(\frac{180}{n}\right)^{\circ}$.

Self-Test 1

Find the area of each polygon.

1. A square with diagonal $9\sqrt{2}$
2. A rectangle with base 12 and diagonal 13
3. A parallelogram with sides 8 and 10 and an angle of measure 60
4. An equilateral triangle with perimeter 12 cm
5. An isosceles triangle with sides 7 cm, 7 cm, and 12 cm
6. A rhombus with diagonals 8 and 10
7. An isosceles trapezoid with legs 5 and bases 9 and 17
8. A regular hexagon with sides 10
9. A regular decagon with sides x and apothem y
10. The quadrilateral shown at the right



Ex. 10

◆ Calculator Key-In

If a regular polygon with n sides is inscribed in a circle with radius 1, then its perimeter and area are given by the formulas derived in Exercise 22 on the preceding page.

$$\text{Perimeter} = 2n \cdot \sin\left(\frac{180}{n}\right)^{\circ} \qquad \text{Area} = n \cdot \sin\left(\frac{180}{n}\right)^{\circ} \cdot \cos\left(\frac{180}{n}\right)^{\circ}$$

Exercises

1. Use the formulas and a calculator to complete the table at the right.

2. Use your answers in Exercise 1 to suggest approximations to the perimeter and the area of a *circle* with radius 1.

Number of sides	Perimeter	Area
18	?	?
180	?	?
1800	?	?
18000	?	?

Circles, Similar Figures, and Geometric Probability

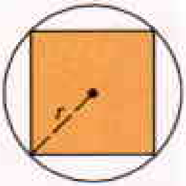
Objectives

1. Know and use the formulas for the circumferences and areas of circles.
2. Know and use the formulas for arc lengths and the areas of sectors of a circle.
3. Find the ratio of the areas of two triangles.
4. Understand and apply the relationships between scale factors, perimeters, and areas of similar figures.
5. Use lengths and areas to solve problems involving geometric probability.

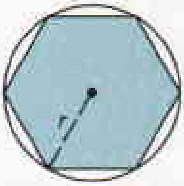
11-5 *Circumferences and Areas of Circles*

When you think of the perimeter of a figure, you probably think of the distance around the figure. Since the word ''around'' is not mathematically precise, perimeter is usually defined in other ways. For example, the perimeter of a polygon is defined as the sum of the lengths of its sides. Since a circle is not a polygon, the perimeter of a circle must be defined differently.

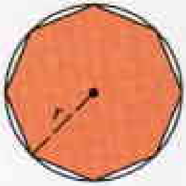
First consider a sequence of regular polygons inscribed in a circle with radius r. Four such polygons are shown below. Imagine that the number of sides of the regular polygons continues to increase. As you can see in the diagrams, the more sides a regular polygon has, the closer it approximates (or "fits") the curve of the circle.



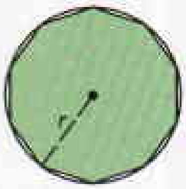
4 sides



6 sides



8 sides



10 sides

Now consider the perimeters and the areas of this sequence of regular polygons. The table below contains values that are approximations (using trigonometry) of the perimeters and the areas of regular polygons in terms of the radius, r.

As the table suggests, these perimeters give us a sequence of numbers that get closer and closer to a limiting number. This limiting number is defined to be the perimeter, or **circumference,** of the circle.

The area of a circle is defined in a similar way. The areas of the inscribed regular polygons get closer and closer to a limiting number, defined to be the **area** of the circle.

The results in the table suggest that the circumference and the area of a circle with radius r are *approximately* $6.28r$ and $3.14r^2$. The exact values are given by the formulas below. (Proofs are suggested in Classroom Exercises 13 and 14 and Written Exercise 33.)

Number of Sides of Polygon	Perimeter	Area
4	$5.66r$	$2.00r^2$
6	$6.00r$	$2.60r^2$
8	$6.12r$	$2.83r^2$
10	$6.18r$	$2.93r^2$
20	$6.26r$	$3.09r^2$
30	$6.27r$	$3.12r^2$
100	$6.28r$	$3.14r^2$

Circumference, C, of circle with radius r: $C = 2\pi r$

Circumference, C, of circle with diameter d: $C = \pi d$

Area, A, of circle with radius r: $A = \pi r^2$

These formulas involve a famous number denoted by the Greek letter π (*pi*), which is the first letter in a Greek word that means "measure around." The number π is the ratio of the circumference of a circle to the diameter. This ratio is a constant for *all* circles. Because π is an irrational number, there isn't any decimal or fraction that expresses the constant number π exactly. Here are some common approximations for π:

$$3.14 \qquad \frac{22}{7} \qquad 3.1416 \qquad 3.14159$$

When you calculate the circumference and area of a circle, leave your answers in terms of π unless you are told to replace π by an approximation.

Example 1 Find the circumference and area of a circle with radius 6 cm.

Solution $C = 2\pi r = 2\pi \cdot 6 = 12\pi$ (cm)
$A = \pi r^2 = \pi \cdot 6^2 = 36\pi$ (cm^2)

Example 2 The photograph shows land that is supplied with water by an irrigation system. This system consists of a moving arm that sprinkles water over a circular region. If the arm is 430 m long, what is the area, correct to the nearest thousand square meters, of the irrigated region? (Use $\pi \approx 3.14$.)

Solution $A = \pi r^2 = \pi \cdot 430^2$
$A \approx 3.14 \cdot 184{,}900 = 580{,}586$
$A \approx 581{,}000$ m^2 (to the nearest 1000 m^2)

Example 3 Find the circumference of a circle if the area is 25π.

Solution Since $\pi r^2 = 25\pi$, $r^2 = 25$ and $r = 5$.
Then $C = 2\pi r = 2\pi \cdot 5 = 10\pi$.

Classroom Exercises

Complete the table. Leave answers in terms of π.

	1.	2.	3.	4.	5.	6.	7.	8.
Radius	3	4	0.8	?	?	?	?	?
Circumference	?	?	?	10π	18π	?	?	?
Area	?	?	?	?	?	36π	49π	144π

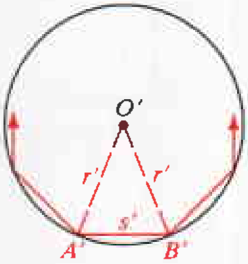
Find the circumference and area to the nearest tenth. Use $\pi \approx 3.14$.

9. $r = 2$ **10.** $r = 6$ **11.** $r = \frac{3}{2}$ **12.** $r = 1.2$

13. The number π is defined to be the ratio of the circumference of a circle to the diameter. This ratio is the same for all circles. Supply the missing reasons for the key steps of proof below.

Given: $\odot O$ and $\odot O'$ with circumferences
C and C' and diameters d and d'

Prove: $\frac{C}{d} = \frac{C'}{d'}$



Key steps of proof:

1. Inscribe in each circle a regular polygon of n sides. Let p and p' be the perimeters.
2. $p = ns$ and $p' = ns'$ (Why?)
3. $\frac{p}{p'} = \frac{ns}{ns'} = \frac{s}{s'}$ (Why?)
4. $\triangle AOB \sim \triangle A'O'B'$ (Why?)
5. $\frac{s}{s'} = \frac{r}{r'} = \frac{d}{d'}$ (Why?)
6. Thus $\frac{p}{p'} = \frac{d}{d'}$. (Steps 3 and 5)
7. Steps 2–5 hold for any number of sides n. We can let n be so large that p is practically the same as C, and p' is practically the same as C'. In advanced courses, you learn that C and C' can be substituted for p and p' in Step 6. This gives $\frac{C}{C'} = \frac{d}{d'}$, or $\frac{C}{d} = \frac{C'}{d'}$.

   (This constant ratio is the number π. Then, since $\frac{C}{d} = \pi$, $C = \pi d$.)

14. Use the formula $C = \pi d$ to derive the formula $C = 2\pi r$.

Written Exercises

Complete the table. Leave answers in terms of π.

A

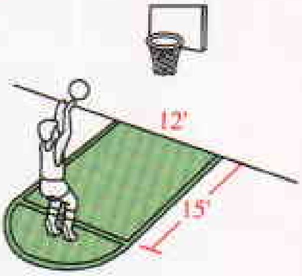
	1.	2.	3.	4.	5.	6.	7.	8.
Radius	7	120	$\frac{5}{2}$	$6\sqrt{2}$	?	?	?	?
Circumference	?	?	?	?	20π	12π	?	?
Area	?	?	?	?	?	?	25π	50π

9. Use $\pi \approx \frac{22}{7}$ to find the circumference and area of a circle when the diameter is **(a)** 42 and **(b)** $14k$.

10. Use $\pi \approx 3.14$ to find the circumference and area of a circle when the diameter is **(a)** 8 (Answer to the nearest tenth.) and **(b)** $4t$.

11. A basketball rim has diameter 18 in. Find the circumference of the rim and the area it encloses. Use $\pi \approx 3.14$.

12. When a basketball player is shooting a free throw, the other players must stay out of the shaded region shown. This region consists of a semicircle together with a rectangle. Find the area of this region to the nearest square foot (ft^2). Use $\pi \approx 3.14$.



13. If 6 oz of dough are needed to make an 8-in. pizza, how much dough will be needed to make a 16-in. pizza of the same thickness? (*Hint*: Compare the areas of the pizza tops.)

14. One can of pumpkin pie mix will make a pie of diameter 8 in. If two cans of pie mix are used to make a larger pie of the same thickness, find the diameter of that pie. Use $\sqrt{2} \approx 1.414$.

15. A school's wrestling mat is a square with 40 ft sides. A circle 28 ft in diameter is painted on the mat. No wrestling is allowed outside the circle. Find the area of the part of the mat that is *not* used for wrestling. Use $\pi \approx \frac{22}{7}$.

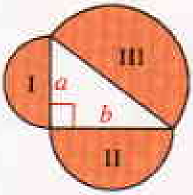
16. An advertisement states that a Roto-Sprinkler can water a circular region with area 1000 ft^2. Find the diameter of this region to the nearest foot. Use $\pi \approx 3.14$.

17. Which is the better buy, a 10-in. pizza costing $5 or a 15-in. pizza costing $9? Use $\pi \approx 3.14$.

18. Semicircles are constructed on the sides of the right triangle shown at the right. If $a = 6$ and $b = 8$, show that

$$\text{Area I} + \text{Area II} = \text{Area III}.$$

B **19.** Repeat Exercise 18 if the right triangle has legs a and b and hypotenuse c.



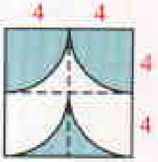
Exs. 18, 19

20. A Ferris wheel has diameter 42 ft. How far will a rider travel during a 4-min ride if the wheel rotates once every 20 seconds? Use $\pi \approx \frac{22}{7}$.

21. The tires of a racing bike are approximately 70 cm in diameter.
- **a.** How far does a bike racer travel in 5 min if the wheels are turning at a speed of 3 revolutions per second? Use $\pi \approx \frac{22}{7}$.
- **b.** How many revolutions does a wheel make in a 22 km race? Use $\pi \approx \frac{22}{7}$.

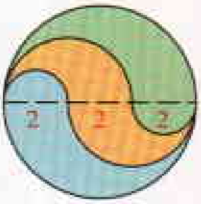
22. A slide projector casts a circle of light with radius 2 ft on a screen that is 10 ft from the projector. If the screen is removed, the projector shines an even larger circle of light on the wall that was 10 ft behind the screen. Find the circumferences and areas of the circles of light on the screen and on the wall. Leave answers in terms of π.

23. A target consists of four concentric circles with radii 1, 2, 3, and 4.
   a. Find the area of the bull's eye and of each ring of the target.
   b. Find the area of the nth ring if the target contains n rings and a bull's eye.



Ex. 24

24. The shaded region in the diagram at the right above is formed by drawing four quarter-circles within a square of side 8. Find the area of the shaded region. (*Hint*: It is possible to give the answer without using pencil and paper or a calculator.)

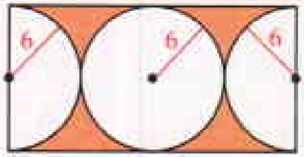


Ex. 25

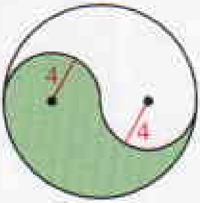
25. The figure at the right consists of semicircles within a circle. Find the area of each shaded region.

Find the area of each shaded region. In Exercise 28, leave your answer in terms of *r*.

26.



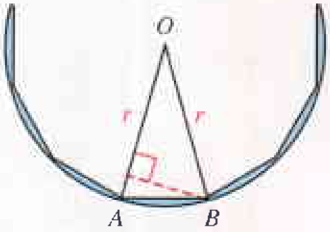
27.



28.

29. Draw a square and its inscribed and circumscribed circles. Find the ratio of the areas of these two circles.

30. Draw an equilateral triangle and its inscribed and circumscribed circles. Find the ratio of the areas of these two circles.



Ex. 31

C **31.** The diagram shows part of a regular polygon of 12 sides inscribed in a circle with radius r. Find the area enclosed between the circle and the polygon in terms of r. Use $\pi \approx 3.14$.

32. A regular octagon is inscribed in a circle with radius r. Find the area enclosed between the circle and the octagon in terms of r. Use $\pi \approx 3.14$ and $\sqrt{2} \approx 1.414$.

33. A regular polygon with apothem a is inscribed in a circle with radius r.
   a. Complete: As the number of sides increases, the value of a gets nearer to $\underline{\ ?\ }$ and the perimeter of the polygon gets nearer to $2\pi r$.
   b. In the formula $A = \frac{1}{2}ap$, replace a by r, and p by $2\pi r$. What formula do you get?

34. Find the circumference of a circle inscribed in a rhombus with diagonals 12 cm and 16 cm.

35. Draw any circle O and any circle P. Construct a circle whose area equals the sum of the areas of circle O and circle P.

♦ Calculator Key-In

The number π is an irrational number. It cannot be expressed exactly as the ratio of two integers. Decimal *approximations* of π have been computed to thousands of decimal places. We can easily look up values in reference books, but such was not always the case. In the past, mathematicians had to rely on their cleverness to compute an approximate value of π. One of the earliest approximations was that of Archimedes, who found that $3\frac{1}{7} > \pi > 3\frac{10}{71}$.

Exercises

1. Find decimal approximations of $3\frac{1}{7}$ and $3\frac{10}{71}$. Did Archimedes approximate π correct to hundredths?

In Exercises 2–4, find approximations for π. The more terms or factors you use, the better your approximations will be.

2. $\pi \approx 2\sqrt{3}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \frac{1}{3^5 \cdot 11} + \cdots\right)$

(Sharpe, 18th century)

3. $\pi \approx 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \cdots$ (Wallis, 17th century)

4. This exercise is for calculators that have a square root function and a memory.

$$\pi \approx 2 \div \left(\sqrt{0.5} \cdot \sqrt{0.5 + 0.5\sqrt{0.5}} \cdot \sqrt{0.5 + 0.5\sqrt{0.5 + 0.5\sqrt{0.5}}} \cdot \ldots\right)$$

(Vieta, 16th century)

Algebra Review: *Evaluating Expressions*

Find the value of each expression using the given values of the variables.

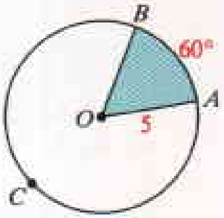
Example $2\pi r$ when $r = \frac{5}{4}$

Solution $2 \cdot \pi \cdot \frac{5}{4} = \left(2 \cdot \frac{5}{4}\right)\pi = \frac{5}{2}\pi$

1. πr^2 when $r = \frac{2}{3}\sqrt{3}$
2. πrl when $r = 4\frac{1}{5}$ and $l = 15$
3. $\frac{1}{3}\pi r^2 h$ when $r = 2\sqrt{6}$ and $h = 4$
4. $\frac{4}{3}\pi r^3$ when $r = 6$
5. $\pi r\sqrt{r^2 + h^2}$ when $r = h = \sqrt{5}$
6. $2\pi r^2 + 2\pi rh$ when $r = 10$ and $h = 6$
7. $\pi r^2 + \pi r\sqrt{r^2 + h^2}$ when $r = 2$ and $h = 2\sqrt{3}$
8. $\pi({r_1}^2 - {r_2}^2)$ when $r_1 = 6$ and $r_2 = 3\sqrt{2}$

11-6 *Arc Lengths and Areas of Sectors*

A *pie chart* is often used to analyze data or to help plan business strategy. The radii of a pie chart divide the interior of the circle into regions called sectors, whose areas represent the relative sizes of particular items. A **sector of a circle** is a region bounded by two radii and an arc of the circle. The shaded region of the diagram at the right below is called sector AOB. The unshaded region is also a sector.



The length of $\overset{\frown}{AB}$ in circle O is part of the circumference of the circle. Since $m\overset{\frown}{AB} = 60$ and $\frac{60}{360} = \frac{1}{6}$, the length of $\overset{\frown}{AB}$ is $\frac{1}{6}$ of the circumference. Thus,

$$\text{Length of } \overset{\frown}{AB} = \frac{1}{6}(2\pi \cdot 5) = \frac{5}{3}\pi.$$

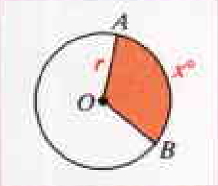
Similarly, the area of sector AOB is $\frac{1}{6}$ of the area of the circle. Thus,

$$\text{Area of sector } AOB = \frac{1}{6}(\pi \cdot 5^2) = \frac{25}{6}\pi.$$

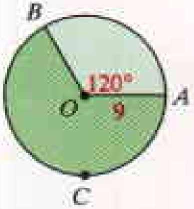
In general, if $m\overset{\frown}{AB} = x$:

$$\text{Length of } \overset{\frown}{AB} = \frac{x}{360} \cdot 2\pi r$$

$$\text{Area of sector } AOB = \frac{x}{360} \cdot \pi r^2$$



Example 1 In $\odot O$ with radius 9, $m\angle AOB = 120$. Find the lengths of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{ACB}$ and the areas of the two sectors shown.



Solution $m\overset{\frown}{AB} = 120$, and $m\overset{\frown}{ACB} = 240$.

Minor arc $\overset{\frown}{AB}$:

$$\text{Length of } \overset{\frown}{AB} = \frac{120}{360} \cdot (2\pi \cdot 9) = \frac{1}{3}(18\pi) = 6\pi$$

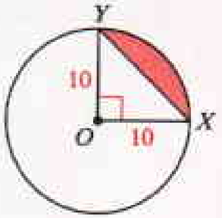
$$\text{Area of sector } AOB = \frac{120}{360} \cdot (\pi \cdot 9^2) = \frac{1}{3}(81\pi) = 27\pi$$

Major arc $\overset{\frown}{ACB}$:

$$\text{Length of } \overset{\frown}{ACB} = \frac{240}{360} \cdot (2\pi \cdot 9) = \frac{2}{3}(18\pi) = 12\pi$$

$$\text{Area of sector} = \frac{240}{360} \cdot (\pi \cdot 9^2) = \frac{2}{3}(81\pi) = 54\pi$$

Example 2 Find the area of the shaded region bounded by $\overline{XY}$ and $\overset{\frown}{XY}$.



Solution

$$\text{Area of sector } XOY = \frac{90}{360} \cdot \pi \cdot 10^2 = 25\pi$$

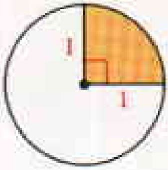
$$\text{Area of } \triangle XOY = \frac{1}{2} \cdot 10 \cdot 10 = 50$$

$$\text{Area of shaded region} = 25\pi - 50$$

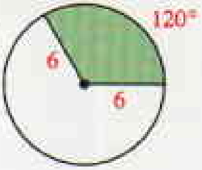
Classroom Exercises

Find the arc length and area of each shaded sector.

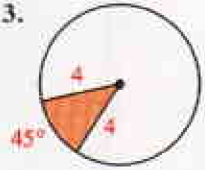
1.



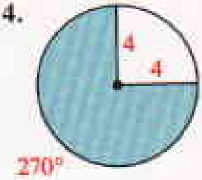
2.



3.



4.



5. In a circle with radius 6, $m\overset{\frown}{AB} = 60$. Make a sketch and find the area of the region bounded by $\overline{AB}$ and $\overset{\frown}{AB}$.

6. A circle has area 160π cm^2. If a sector of the circle has area 40π cm^2, find the measure of the arc of the sector.

7. Compare the areas of two sectors if
  a. they have the same central angle, but the radius of one is twice as long as the radius of the other.
  b. they have the same radius, but the central angle of one is twice as large as the central angle of the other.

Written Exercises

Sector *AOB* is described by giving $m\angle AOB$ and the radius of circle *O*. Make a sketch and find the length of $\overset{\frown}{AB}$ and the area of sector *AOB*.

A

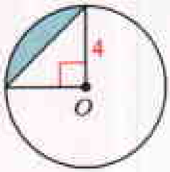
	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
$m\angle AOB$	30	45	120	240	180	270	40	320	108	192
radius	12	4	3	3	1.5	0.8	$\frac{9}{2}$	$1\frac{1}{5}$	$5\sqrt{2}$	$3\sqrt{3}$

11. The area of sector *AOB* is 10π and $m\angle AOB = 100$. Find the radius of circle *O*.

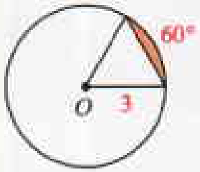
12. The area of sector *AOB* is $\frac{7\pi}{2}$ and $m\angle AOB = 315$. Find the radius of circle *O*.

Find the area of each shaded region. Point *O* marks the center of a circle.

B **13.**



14.



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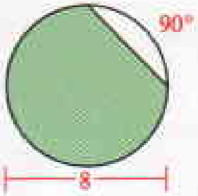
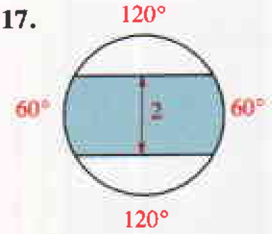
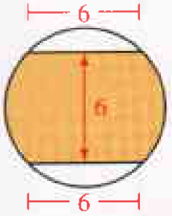


16.

17.



18.



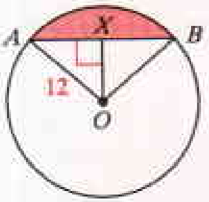
19. A rectangle with length 16 cm and width 12 cm is inscribed in a circle. Find the area of the region inside the circle but outside the rectangle.

20. From point P, $\overline{PA}$ and $\overline{PB}$ are drawn tangent to circle O at points A and B. If the radius of the circle is 6 and $m\angle APB = 60$, find the area of the region outside the circle but inside quadrilateral $AOBP$.

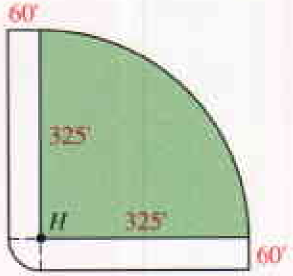
You may wish to use a calculator for Exercises 21–23. Use $\pi \approx 3.14$.

21. Chord AB is 18 cm long and the radius of the circle is 12 cm.

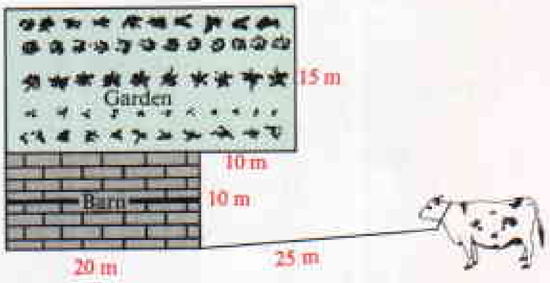
- **a.** Use trigonometry to find the measures of $\angle AOX$ and $\angle AOB$, correct to the nearest integer.
- **b.** Find the area of the shaded region to the nearest square centimeter. Use $\sqrt{7} \approx 2.646$.



22. The diagram shows some dimensions in a baseball stadium. H represents home plate. Approximate the ratio of the areas of fair territory (shaded region) and foul territory (nonshaded region).

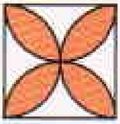


23. A cow is tied by a 25 m rope to the corner of a barn as shown. A fence keeps the cow out of the garden. Find, to the nearest square meter, the grazing area. Use $\sqrt{2} \approx 1.414$.

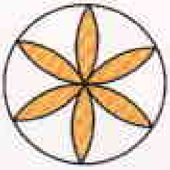


24. *ABCD* is a square with sides 8 cm long. Two circles each with radius 8 cm are drawn, one with center *A* and the other with center *C*. Find the area of the region inside both circles.

25. Two circles have radii 6 cm and their centers are 6 cm apart. Find the area of the region common to both circles.



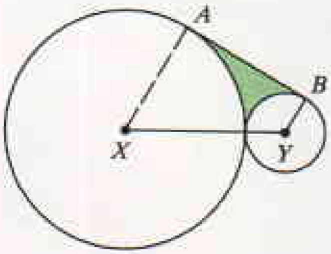
26. **a.** Draw a square. Then construct the figure shown at the right.
b. If the radius of the square is 2, find the area of the shaded region.



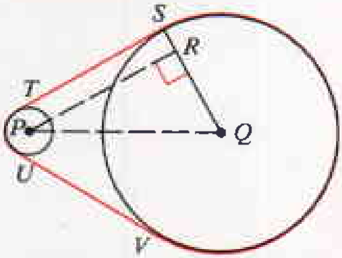
C **27.** **a.** Using only a compass, construct the six-pointed figure shown at the right.
b. If the radius of the circle is 6, find the area of the shaded region.

28. Three circles with radii 6 are tangent to each other. Find the area of the region enclosed between them.

★ **29.** Circles *X* and *Y*, with radii 6 and 2, are tangent to each other. $\overline{AB}$ is a common external tangent. Find the area of the shaded region. (*Hint*: What kind of figure is *AXYB*? What is the measure of $\angle AXY$?)



Ex. 29



Ex. 30

★ **30.** The diagram at the right above shows a belt tightly stretched over two wheels with radii 5 cm and 25 cm. The distance between the centers of the wheels is 40 cm. Find the length of the belt.

Challenge

Here $\overline{XY}$ has been divided into five congruent segments and semicircles have been drawn. But suppose $\overline{XY}$ were divided into millions of congruent segments and semicircles were drawn. What would the sum of the lengths of the arcs be?

Sarah says, "*XY*, because all the points would be so close to $\overline{XY}$." Mike says, "A really large number, because there would be so many arc lengths to add up." What do you say?

11-7 *Ratios of Areas*

In this section you will learn to compare the areas of figures by finding ratios.

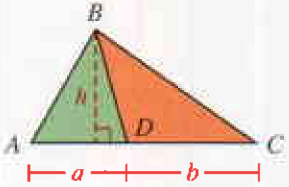
Example 1 Find the ratios of the areas of two triangles:

a. with equal heights
b. with equal bases
c. that are similar

Solution

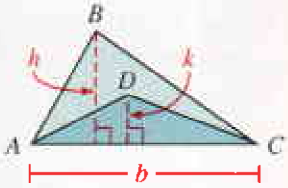
a. $$\frac{\text{area of } \triangle ABD}{\text{area of } \triangle DBC} = \frac{\frac{1}{2}ah}{\frac{1}{2}bh} = \frac{a}{b}$$

ratio of areas = ratio of bases



b. $$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle ADC} = \frac{\frac{1}{2}bh}{\frac{1}{2}bk} = \frac{h}{k}$$

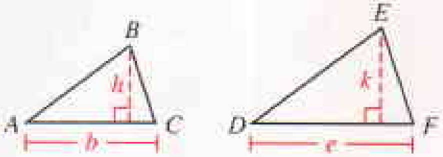
ratio of areas = ratio of heights



c. $$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEF} = \frac{\frac{1}{2}bh}{\frac{1}{2}ek} = \frac{bh}{ek} = \frac{b}{e} \cdot \frac{h}{k}$$

It follows from Exercise 25 on page 259 that if

$\triangle ABC \sim \triangle DEF$, then $\frac{h}{k} = \frac{b}{e}$.



$\triangle ABC \sim \triangle DEF$

$$\text{Thus, } \frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEF} = \frac{b}{e} \cdot \frac{h}{k} = \frac{b}{e} \cdot \frac{b}{e} = \left(\frac{b}{e}\right)^2 = (\text{scale factor})^2.$$

ratio of areas = square of scale factor

Example 1 justifies the following properties.

Comparing Areas of Triangles

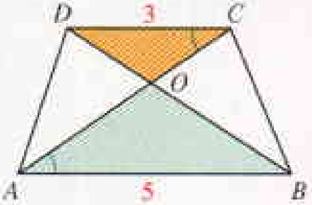
1. If two triangles have equal heights, then the ratio of their areas equals the ratio of their bases.
2. If two triangles have equal bases, then the ratio of their areas equals the ratio of their heights.
3. If two triangles are similar, then the ratio of their areas equals the square of their scale factor.

Example 2 *ABCD* is a trapezoid. Find the ratio of the areas of:

a. $\triangle COD$ and $\triangle AOB$

b. $\triangle COD$ and $\triangle AOD$

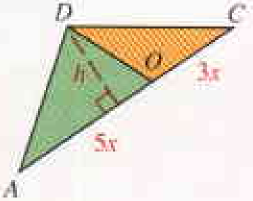
c. $\triangle OAB$ and $\triangle DAB$



Solution $\triangle COD \sim \triangle AOB$ by the AA Similarity Postulate, with a scale factor of 3:5. Thus each of the corresponding sides and heights of these triangles has a 3:5 ratio.

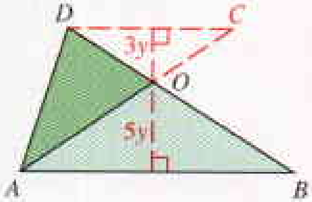
a. Since $\triangle COD \sim \triangle AOB$,

$$\frac{\text{area of } \triangle COD}{\text{area of } \triangle AOB} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}.$$



b. Since $\triangle COD$ and $\triangle AOD$ have the same height, h, their area ratio equals their base ratio.

$$\frac{\text{area of } \triangle COD}{\text{area of } \triangle AOD} = \frac{CO}{AO} = \frac{3x}{5x} = \frac{3}{5}$$



c. Since $\triangle OAB$ and $\triangle DAB$ have the same base, $\overline{AB}$, their area ratio equals their height ratio. Notice that the height of $\triangle DAB$ is $3y + 5y$, or $8y$.

$$\frac{\text{area of } \triangle OAB}{\text{area of } \triangle DAB} = \frac{5y}{8y} = \frac{5}{8}$$

You know that the ratios of the perimeters and areas of two similar triangles are related to their scale factor. These relationships can be generalized to any two similar figures.

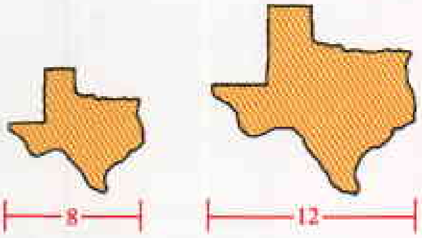
Theorem 11-7

If the scale factor of two similar figures is $a:b$, then

(1) the ratio of the perimeters is $a:b$.

(2) the ratio of the areas is $a^2:b^2$.

Example 3 Find the ratio of the perimeters and the ratio of the areas of the two similar figures.

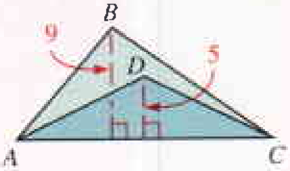


Solution The scale factor is 8:12, or 2:3. Therefore, the ratio of the perimeters is 2:3. The ratio of the areas is $2^2:3^2$, or 4:9.

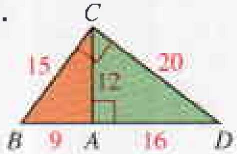
Classroom Exercises

Find the ratio of the areas of $\triangle ABC$ and $\triangle ADC$.

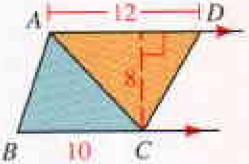
1.



2.



3.

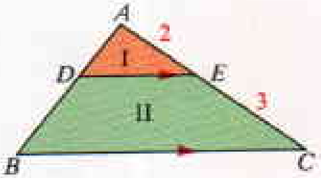


The table refers to similar figures. Complete the table.

	4.	5.	6.	7.	8.	9.	10.	11.
Scale factor	1:3	1:5	3:4	2:3	?	?	?	?
Ratio of perimeters	?	?	?	?	4:5	3:5	?	?
Ratio of areas	?	?	?	?	?	?	16:49	36:25

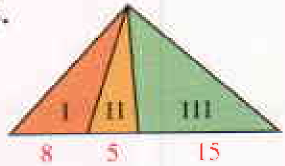
12. a. Are all circles similar?
   b. If two circles have radii 9 and 12, what is the ratio of the circumferences? of the areas?

13. a. Are regions I and II similar?
   b. Name two similar triangles.
   c. What is the ratio of their areas?
   d. What is the ratio of the areas of regions I and II?



Find the ratio of the areas of triangles (a) I and II and (b) I and III.

14.



15.

Written Exercises

The table refers to similar figures. Copy and complete the table.

A

	1.	2.	3.	4.	5.	6.	7.	8.
Scale factor	1:4	3:2	$r:2s$	?	?	?	?	?
Ratio of perimeters	?	?	?	9:5	3:13	?	?	?
Ratio of areas	?	?	?	?	?	25:1	9:64	2:1

9. On a map of California, 1 cm corresponds to 50 km. Find the ratio of the map's area to the actual area of California.

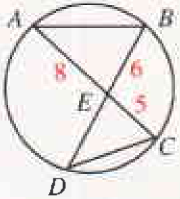
10. The areas of two circles are 36π and 64π. What is the ratio of the diameters? of the circumferences?

11. L, M, and N are the midpoints of the sides of $\triangle ABC$. Find the ratio of the perimeters and the ratio of the areas of $\triangle LMN$ and $\triangle ABC$.

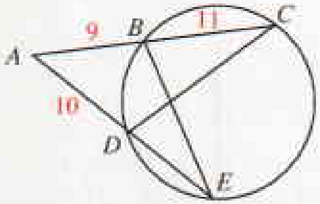
12. The lengths of two similar rectangles are x^2 and xy, respectively. What is the ratio of the areas?

Name two similar triangles and find the ratio of their areas. Then find *DE*.

13.



14.

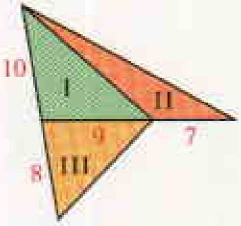


15. A quadrilateral with sides 8 cm, 9 cm, 6 cm, and 5 cm has area 45 cm^2. Find the area of a similar quadrilateral whose longest side is 15 cm.

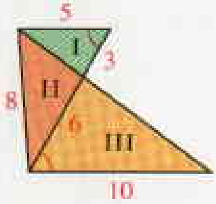
16. A pentagon with sides 3 m, 4 m, 5 m, 6 m, and 7 m has area 48 m^2. Find the perimeter of a similar pentagon whose area is 27 m^2.

Find the ratio of the areas of triangles (a) I and II and (b) I and III. In Exercise 19(b), use the fact that Area I + Area II = Area III.

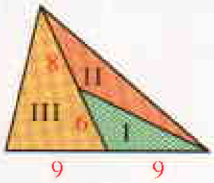
B **17.**



18.

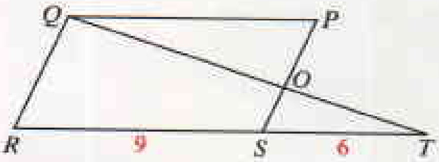


19.

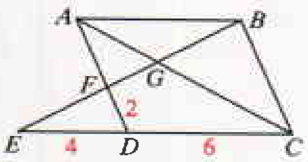


20. In the diagram below, $PQRS$ is a parallelogram. Find the ratio of the areas for each pair of triangles.

a. $\triangle TOS$ and $\triangle QOP$ **b.** $\triangle TOS$ and $\triangle TQR$



Ex. 20



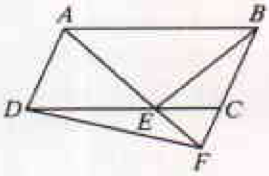
Ex. 21

21. In the diagram above, $ABCD$ is a parallelogram. Name *four pairs* of similar triangles and give the ratio of the areas for each pair.

22. The area of parallelogram $ABCD$ is 48 cm^2 and $DE = 2 \cdot EC$. Find the area of:

a. $\triangle ABE$ **b.** $\triangle BEC$ **c.** $\triangle ADE$

d. $\triangle CEF$ **e.** $\triangle DEF$ **f.** $\triangle BEF$

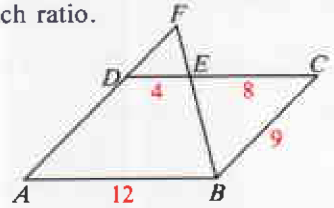


23. $ABCD$ is a parallelogram. Find each ratio.

a. $\dfrac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABF}$

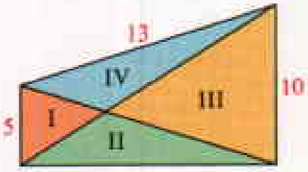
b. $\dfrac{\text{Area of } \triangle DEF}{\text{Area of } \triangle CEB}$

c. $\dfrac{\text{Area of } \triangle DEF}{\text{Area of trap. } DEBA}$

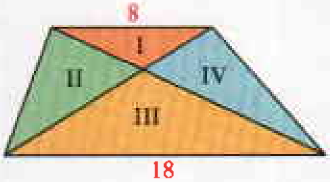


The figures in Exercises 24 and 25 are trapezoids. Find the ratio of the areas of (a) △I and △III, (b) △I and △II, (c) △I and △IV, (d) △II and △IV, and (e) △I and the trapezoid.

24.

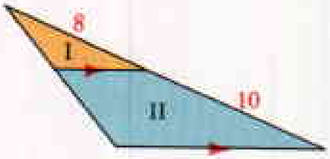


25.

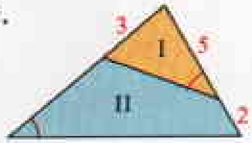


Find the ratio of the areas of regions I and II.

26.



27.



28.

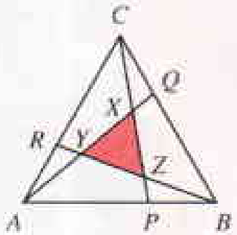
C **29.** G is the intersection point of the medians of $\triangle ABC$. A line through G parallel to $\overline{BC}$ divides the triangle into two regions. What is the ratio of their areas? (*Hint*: See Theorem 10-4, page 387.)

30. In $\triangle LMN$, altitude $\overline{LK}$ is 12 cm long. Through point J of $\overline{LK}$ a line is drawn parallel to $\overline{MN}$, dividing the triangle into two regions with equal areas. Find LJ.

31. If you draw the three medians of a triangle, six small triangles are formed. Prove whatever you can about the areas of these six triangles.

★ **32.** $\triangle ABC$ is equilateral; $\dfrac{AP}{PB} = \dfrac{BQ}{QC} = \dfrac{CR}{RA} = \dfrac{2}{1}$.

Prove: Area of $\triangle XYZ = \frac{1}{7}$(area of $\triangle ABC$)

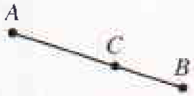


Ex. 32

11-8 *Geometric Probability*

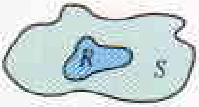
The geometric probability problems in this section can be solved by using one of the following two principles.

1. Suppose a point P of $\overline{AB}$ is picked at random. Then:



$$\text{probability that } P \text{ is on } \overline{AC} = \frac{\text{length of } \overline{AC}}{\text{length of } \overline{AB}}$$

2. Suppose a point P of region S is picked at random. Then:



$$\text{probability that } P \text{ is in region } R = \frac{\text{area of } R}{\text{area of } S}$$

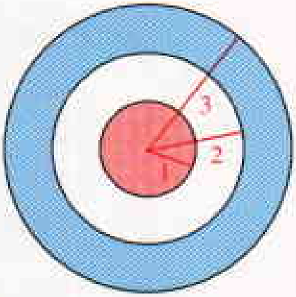
Example 1 Every ten minutes a bus pulls up to a hotel and waits for two minutes while passengers get on and off. Then the bus leaves. If a person walks out of the hotel front door at a random time, what is the probability that a bus is there?

Solution Think of a time line in which the colored segments represent times when the bus is at the hotel. For *any* ten-minute period, a two-minute subinterval is colored. Thus:

$$\text{probability that a bus will be there} = \frac{\text{length of colored segment}}{\text{length of whole segment}}$$

$$= \frac{2}{10} = \frac{1}{5}$$

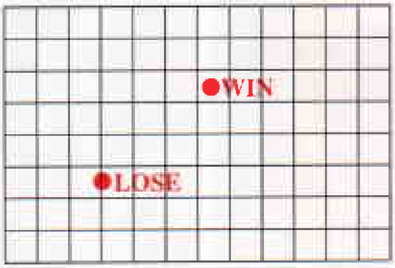
Example 2 A person who is just beginning archery lessons misses the target frequently. And when a beginner hits the target, each spot is as likely to be hit as another. If a beginner shoots an arrow and it hits the target, what is the probability that the arrow hits the red bull's eye?



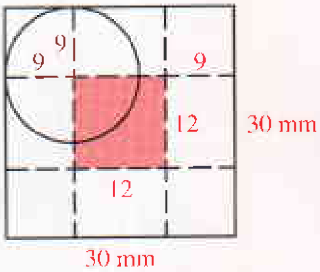
Solution probability arrow hits bull's eye if it hits target =

$$\frac{\text{area of bull's eye}}{\text{area of target}} = \frac{\pi \cdot 1^2}{\pi \cdot 3^2} = \frac{1}{9}$$

Example 3 At a carnival game, you can toss a coin on a large table that has been divided into squares 30 mm on a side. If the coin comes to rest without touching any line, you win. Otherwise you lose your coin. What are your chances of winning on one toss of a dime? (A dime has a radius of 9 mm.)



Solution Although there are many squares on the board, it is only necessary to consider the square in which the center of the dime lands. In order for the dime to avoid touching a line, the dime's center must be more than 9 mm from each side of the square. Its center must land in the shaded square shown. Thus the probability that the dime does not touch a line is equal to the probability that the center of the dime lies in the shaded region.



$$\text{probability of winning} = \frac{\text{area of shaded square}}{\text{area of larger square}} = \frac{12^2}{30^2} = 0.16$$

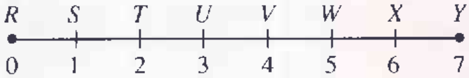
Note: In practice, your probability of winning would be less than 16% for several reasons. For example, a carnival might require you to toss a quarter instead of the smaller dime, as discussed in Written Exercise 9.

Classroom Exercises

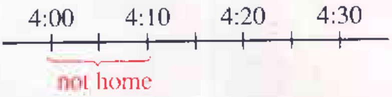
1. A point P is picked at random on $\overline{RW}$. What is the probability that P is on:

a. $\overline{RS}$? **b.** $\overline{SV}$? **c.** $\overline{SW}$?

d. $\overline{RW}$? **e.** $\overline{XY}$? **f.** $\overline{RY}$?

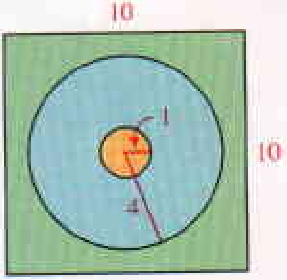


2. A friend promises to call you sometime between 4:00 and 4:30 P.M. If you are not home to receive the call until 4:10, what is the probability that you miss the first call that your friend makes to you?



3. A dart lands at a random point on the square dartboard shown. What is the probability that the dart is within the outer circle? within the bull's eye? Use $\pi \approx 3.14$.

4. A ship is known to have sunk in the ocean in a square region 100 mi on a side. A salvage vessel anchors at a random spot in this square. Divers search 1 mi in all directions from the point on the ocean floor directly below the vessel. What is the approximate probability that they locate the sunken ship on the first try? Use $\pi \approx 3.14$.



Written Exercises

If a value of π is required in the following exercises, use $\pi \approx 3.14$.

A

1. M is the midpoint of $\overline{AB}$ and Q is the midpoint of $\overline{MB}$. If a point of $\overline{AB}$ is picked at random, what is the probability that the point is on $\overline{MQ}$? (*Hint*: Make a sketch.)

2. In the diagram, $AC = CB$, $CD = DB$, and $DE = EB$. If a point X is selected at random from $\overline{AB}$, what is the probability that:

a. X is between A and C?

b. X is between D and B?

c. X is between C and E?

3. A friend promises to call you at home sometime between 3 P.M. and 4 P.M. At 2:45 P.M. you must leave your house unexpectedly for half an hour. What is the probability you miss the first call?

4. a. At a subway stop, a train arrives every six minutes, waits one minute, and then leaves. If you arrive at a random time, what is the probability there will be a train waiting?

b. If you arrive and there is no train waiting, what is the probability that you will wait no more than two minutes before one arrives?

5. A circular dartboard has diameter 40 cm. Its bull's eye has diameter 8 cm.

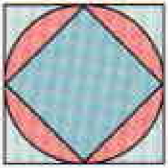
a. If an amateur throws a dart and it hits the board, what is the probability that the dart hits the bull's eye?

b. After many throws, 75 darts have hit the target. Estimate the number hitting the bull's eye.

6. Several hundred darts are thrown at the square dartboard shown. About what percentage of those hitting the board will land in the location described?

a. Inside the inner square

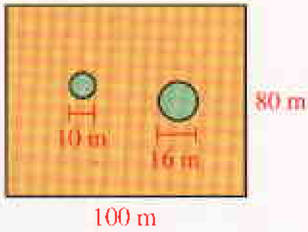
b. Outside the inner square but inside the circle



7. A dart is thrown at a board 12 m long and 5 m wide. Attached to the board are 30 balloons, each with radius 10 cm. Assuming each balloon lies entirely on the board, find the probability that a dart that hits the board will also hit a balloon.

8. Parachutists jump from an airplane and land in the rectangular field shown. What is the probability that a parachutist avoids the two trees represented by circles in the diagram? (Assume that the person is unable to control the landing point.)

9. Refer to Example 3. Suppose that a quarter, instead of a dime, is tossed and lands on the table shown on the preceding page. What is the probability of winning on one toss? (The radius of a quarter is 12 mm.)



10. Repeat Exercise 9, using a nickel instead of a quarter. The radius of a nickel is 11 mm.

B **11.** A piece of wire 6 in. long is cut into two pieces at a random point. What is the probability that both pieces of wire will be at least 1 in. long?

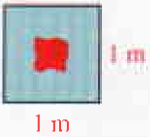
12. A piece of string 8 cm long is cut at a random point. What is the probability that:

a. each piece is at least 2 cm long?

b. the lengths of the two pieces differ by no more than 2 cm?

c. the lengths of the two pieces total 8 cm?

13. Darts are thrown at a 1-meter square which contains an irregular red region. Of 100 darts thrown, 80 hit the square. Of these, 10 hit the red region. Estimate the area of this region.



14. A carnival game has a white dartboard 5 m long and 2 m wide on which 100 red stars are painted. Each player tries to hit a star with a dart. Before trying it, you notice that only 3 shots out of the previous 50 hit a star. Estimate the area of one star.

15. a. Suppose that a coin with radius R is tossed and lands on the table shown on page 462. Show that the probability the coin does not touch a line is $\left(\frac{30 - 2R}{30}\right)^2$.

b. Find the value of R for which the probability is 0.25.

16. A and B are the endpoints of a diameter. If C is a point chosen at random from the points on the circle (excluding A and B), what is the probability that:

a. $\triangle ABC$ is a right triangle? **b.** $m\angle CAB \leq 30$?

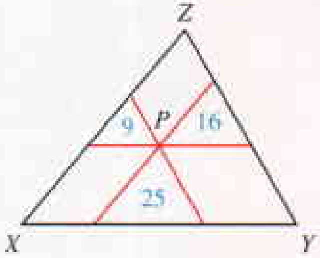
C **17.** A researcher was tape recording birdcalls. The tape recorder had a 1-hour tape in it. Eight minutes after the recorder was turned on, a 5-minute birdcall began. Unfortunately, the researcher accidentally erased 10 min of the tape. What is the probability that:

a. part of the birdcall was erased? **b.** all of the birdcall was erased?

(*Hint*: Draw a time line from 0 to 60 min and locate on this line when the birdcall took place. Also consider the possible starting times when the erasure could have occurred.)

Challenge

Three segments through point P and parallel to the sides of $\triangle XYZ$ divide the whole region into six subregions. The three triangular subregions have the areas shown. Find the area of $\triangle XYZ$.



Self-Test 2

Leave your answers in terms of π unless you are told to use an approximation.

1. Find the circumference and area of a circle with radius 14. Use $\pi \approx \frac{22}{7}$.

2. The circumference of a circle is 18π. What is its area?

3. In $\odot O$ with radius 12, $m\widehat{AB} = 90$.

  a. Find the length of $\widehat{AB}$.

  b. Find the area of sector AOB.

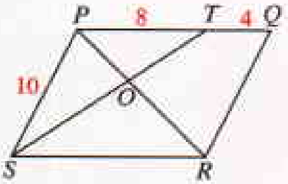
  c. Find the area of the region bounded by $\overline{AB}$ and $\widehat{AB}$.

4. Find the ratio of the areas of two circles with radii 4 and 7.

5. The areas of two similar triangles are 36 and 81. Find the ratio of their perimeters.

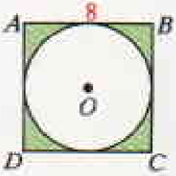
6. $PQRS$ is a parallelogram. Find the ratio of the areas of:

  a. $\triangle PTO$ and $\triangle RSO$ **b.** $\triangle RPS$ and $\triangle TPS$



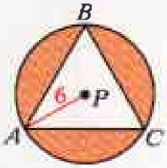
Each polygon is a regular polygon. Find the area of the shaded region.

7.



$\odot O$ is inscribed in square $ABCD$.

8.



$\triangle ABC$ is inscribed in $\odot P$.

9. Refer to $\square PQRS$ in Exercise 6. A point is randomly chosen on $\overline{PR}$. Find the probability that the point is on $\overline{OR}$.

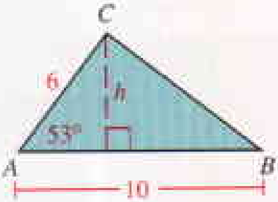
10. Suppose that the figure in Exercise 7 is a dartboard. Imagine that someone with poor aim throws a dart and you *hear* it hit the dartboard. What is the probability that the dart landed inside the circle?

Extra

Congruence and Area

The SAS Postulate tells us that a triangle is *determined*, or fixed in size and shape, when two sides and the included angle are fixed. This means that the other parts of the triangle and its area can be determined from the given SAS information. Similarly, the area of a triangle can be determined when given ASA, SSS, AAS, or HL information. Computing the area of a triangle can often be simplified by using a calculator.

Example 1 Given the SAS information shown for $\triangle ABC$, find its area.

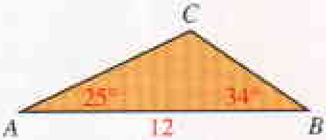


Solution Draw the altitude from C.

Then $\frac{h}{6} = \sin 53° \approx 0.7986$; $h \approx 4.79$.

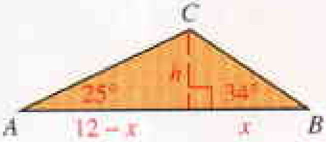
Area $= \frac{1}{2}bh \approx \frac{1}{2}(10)(4.79) \approx 24.0$

Example 2 Given the ASA information shown for $\triangle ABC$, find its area.



Solution

Step 1 Draw the altitude from C.



Then $\tan 25° = \frac{h}{12 - x}$ and $\tan 34° = \frac{h}{x}$.

$$\begin{aligned}
(12 - x)\tan 25° &= h \quad \text{and} \quad x \tan 34° = h \\
(12 - x)\tan 25° &= x \tan 34° \\
(12 - x)(0.4663) &\approx x(0.6745) \\
5.5956 - 0.4663x &\approx 0.6745x \\
5.5956 &\approx 1.1408x \\
4.905 &\approx x
\end{aligned}$$

Step 2 Knowing x, we can find h:

$h = x \tan 34° \approx (4.905)(0.6745) \approx 3.308$

Step 3 Area $= \frac{1}{2}bh \approx \frac{1}{2}(12)(3.308) \approx 19.8$

Exercises

Use the given information to find the approximate area of $\triangle ABC$. In Exercises 6 and 7 the altitude from C lies outside the triangle.

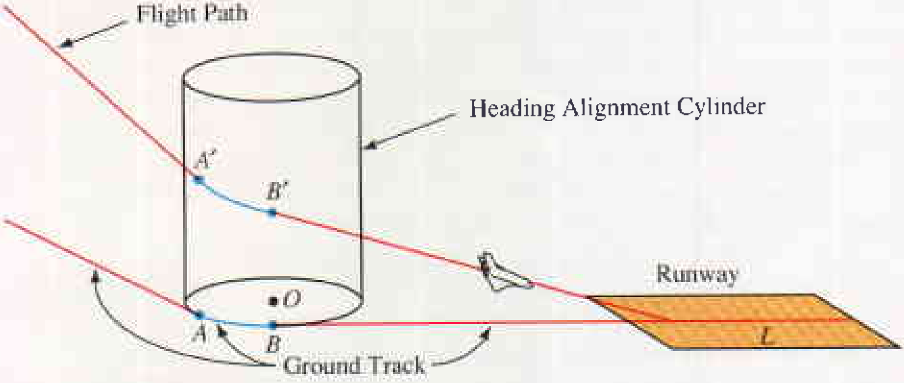
1. (SAS) $AB = 8$, $m\angle B = 67$, $BC = 15$
2. (HL) $m\angle C = 90$, $AB = 30$, $BC = 20$ (Use $\sqrt{5} \approx 2.236$.)
3. (SSS) $AB = 10$, $BC = 12$, $CA = 8$ (*Hint*: Use Heron's Formula.)
4. (ASA) $m\angle A = 28$, $AB = 10$, $m\angle B = 42$
5. (AAS) $m\angle A = 36$, $m\angle B = 80$, $BC = 10$ (*Hint*: Find the measure of $\angle C$. Then proceed as in Example 2.)
6. (SAS) $AB = 12$, $m\angle A = 118$, $AC = 20$
7. (ASA) $m\angle A = 107$, $AB = 20$, $m\angle B = 35$

★ **8.** The two triangles shown have two pairs of congruent corresponding sides and one pair of congruent corresponding non-included angles (SSA). Of course, they are *not* congruent. Find the area of each triangle.

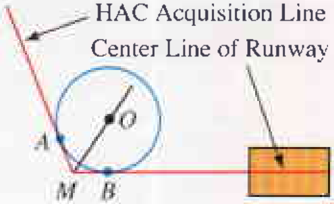
Application

Space Shuttle Landings

The space shuttle is launched vertically as a rocket but lands horizontally as a glider with no power and no second chance at the runway. NASA studied many different guidance systems for the final portion of entry and landing. The system that was selected for the first shuttle flights used a cylinder called the *Heading Alignment Cylinder* (HAC), shown in the diagram below. Notice that the projection of the flight path onto the Earth's surface is called the *ground track*.



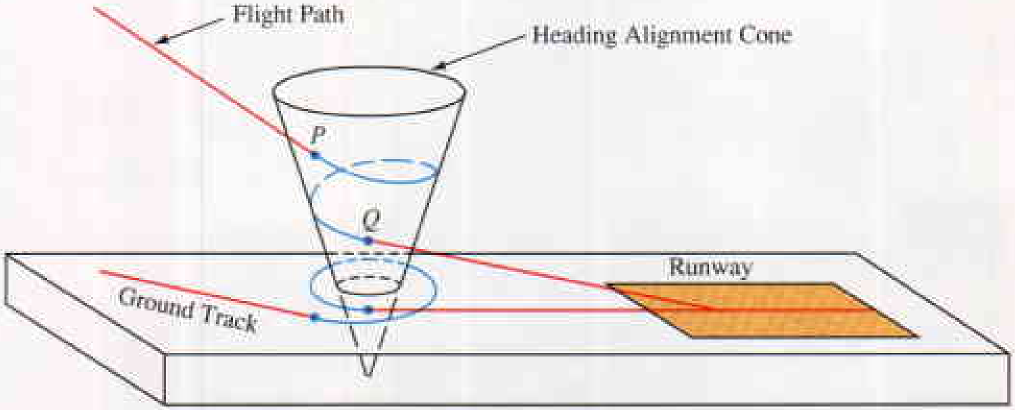
The shuttle followed a straight-line flight path to A' and then it followed a curved path along the Heading Alignment Cylinder to point B'. The shuttle continued to lose altitude so that it was closer to the Earth's surface at B' than at A'. From B' to landing at L the shuttle followed a straight path aligned with the center of the runway.



The points A' and B' where the shuttle's turn begins and ends can be determined by looking at the ground track. The figure at the right shows what you would see if you were high above the ground looking straight down on the ground track and runway. A' and B' are directly above A and B, which are located as follows: Extend the HAC acquisition line and the center line of the runway to meet at M. Bisect the angle at M and choose point O on the bisector so that a circle with center O and radius 20,000 ft will be tangent to the sides of the angle. Call the points of tangency A and B. $\odot O$ is the base of the Heading Alignment Cylinder and A' and B' are on the cylinder directly above A and B.

Normally the shuttle approached the cylinder at 800 ft/s and turned along its surface by lowering one wing tip so that the wings formed an angle of about 45° with the horizontal (called the *bank angle*). Under high-speed conditions the shuttle would have approached the cylinder at 1000 ft/s. This would have required a bank angle of about 57° to follow the surface of the cylinder. Unfortunately, at that bank angle, the shuttle would have lost lift, and the astronauts would have lost some of their control capability.

NASA has refined the guidance system so that the shuttle can be safely landed even under these adverse circumstances. Now, instead of following the surface of a cylinder, it spirals along the surface of a cone called the *Heading Alignment Cone*. Once every second during this part of the landing the shuttle's computers recompute the radius of turn necessary to keep the shuttle on the surface of the cone. Now even under most high-speed conditions the bank angle will not exceed approximately 42°. At point Q the shuttle is heading directly toward the runway; it leaves the cone and continues along a straight course to touchdown.



Exercises

1. Let T be any point on the bisector of $\angle AMB$. Show that if $\odot T$ is drawn tangent to $\overleftrightarrow{MA}$ it will also be tangent to $\overleftrightarrow{MB}$.
2. The radius of the Heading Alignment Cylinder was 20,000 ft, and a typical measure for $\angle AMB$ was 120. How long was the curved portion of the ground track, $\overset{\frown}{AB}$? (Use 3.1416 for π.)
3. The shuttle's turning radius changes as it moves along the surface of the Heading Alignment Cone. Is the radius larger near P or near Q?
4. A good approximation of the detailed landing procedure uses a Heading Alignment Cone with vertex below the surface of the Earth. A typical radius of the cone at a height of 30,000 ft above the Earth's surface is 20,000 ft. At a height of 12,000 ft, which is a typical height for Q, the radius of the cone is 14,000 ft.
   a. How far below the surface of the Earth is the vertex of the cone?
   b. What is the radius of the cone at a height of 15,000 ft?
   c. At what height is the radius of the cone equal to 12,000 ft?

Chapter Summary

1. If two figures are congruent, then they have the same area.
2. The area of a region is the sum of the areas of its non-overlapping parts.
3. The list below gives the formulas for areas of polygons.

   Square: $A = s^2$
   Rectangle: $A = bh$
   Parallelogram: $A = bh$
   Triangle: $A = \frac{1}{2}bh$
   Rhombus: $A = \frac{1}{2}d_1d_2$
   Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$
   Regular polygon: $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter

4. The list below gives the formulas related to circles.

$$\begin{array}{ll} C = 2\pi r & \\ C = \pi d & \\ A = \pi r^2 & \end{array} \qquad \left.\begin{array}{l} \text{Length of arc} = \dfrac{x}{360}\cdot 2\pi r \\ \\ \text{Area of sector} = \dfrac{x}{360}\cdot \pi r^2 \end{array}\right\} \text{ where } x \text{ is the measure of the arc}$$

5. If two triangles have equal heights, then the ratio of their areas equals the ratio of their bases. If two triangles have equal bases, then the ratio of their areas equals the ratio of their heights.
6. If the scale factor of two similar figures is $a:b$, then
   (1) the ratio of the perimeters is $a:b$.
   (2) the ratio of the areas is $a^2:b^2$.
7. Two principles used in geometric probability problems are stated and illustrated on page 461.

Chapter Review

1. Find the area of a square with perimeter 32. **11–1**

2. Find the area of a rectangle with length 4 and diagonal 6.

3. Find the area of a square with side $3\sqrt{2}$ cm.

4. Find the area of a rhombus with side 17 and longer diagonal 30. **11–2**

5. A parallelogram has sides 8 and 12. The shorter altitude is 6. Find the length of the other altitude.

6. Find the perimeter and the area of the triangle shown.

7. Find the height of a trapezoid with median 12 and area 84. **11–3**

8. Find the area of an isosceles trapezoid with legs 5 and bases 4 and 12.

9. Find the perimeter and the area of the figure shown.

10. Find the area of a square with apothem 3 m. **11–4**

11. Find the area of an equilateral triangle with radius $2\sqrt{3}$.

12. Find the area of a regular hexagon with perimeter 12 cm.

13. Find the circumference and area of a circle with radius 30. Use $\pi \approx 3.14$. **11–5**

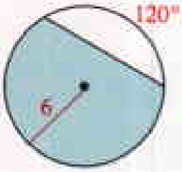
14. The area of a circle is 121π cm^2. Find the diameter.

15. A square with side 8 is inscribed in a circle. Find the circumference and the area of the circle.

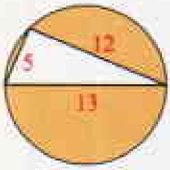
16. Find the length of a 135° arc in a circle with radius 24. **11–6**

Find the area of each shaded region.

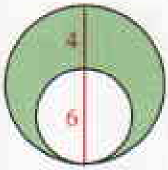
17.



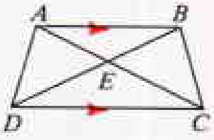
18.



19.



20. If $AB = 9$ and $CD = 12$, find the ratio of the areas of: **11–7**
a. $\triangle AEB$ and $\triangle DEC$ **b.** $\triangle AED$ and $\triangle DEC$



21. Two regular octagons have perimeters 16 cm and 32 cm, respectively. What is the ratio of their areas?

22. Two similar polygons have the scale factor 7:5. The area of the large polygon is 147. Find the area of the smaller polygon.

23. A point is randomly chosen inside the larger circle of Exercise 19. What is the probability that the point is inside the smaller circle? **11–8**

Chapter Test

Find the area of each figure described.

1. A circle with diameter 10

2. A square with diagonal 4 cm

3. An isosceles right triangle with hypotenuse $6\sqrt{2}$

4. A circle with circumference 30π m

5. A rhombus with diagonals 5 and 4

6. An isosceles trapezoid with legs 10 and bases 6 and 22

7. A parallelogram with sides 6 and 10 that form a 30° angle

8. A regular hexagon with apothem $2\sqrt{3}$ cm

9. Sector AOB of $\odot O$ with radius 4 and $m\overset{\frown}{AB} = 45$

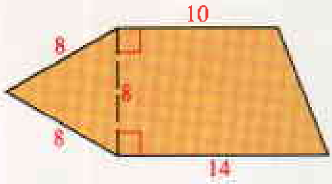
10. A rectangle with length 12 inscribed in a circle with radius 7.5

11. A sector of a circle with radius 12 and arc length 10π

12. A square with radius 9

Find the area of each shaded region.

13.



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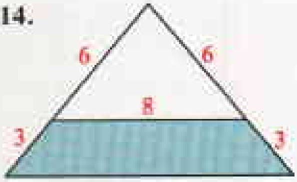

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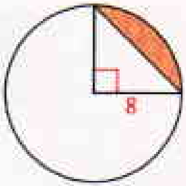
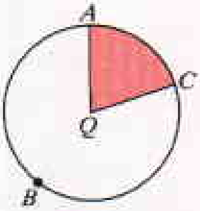


16. The areas of two circles are 100π and 36π. Find the ratio of their radii and the ratio of their circumferences.

17. Two regular pentagons have sides of 14 m and 3.5 m, respectively. Find their scale factor and the ratio of their areas.

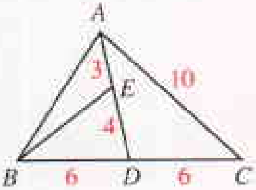
18. In the diagram of $\odot Q$, $m\overset{\frown}{ABC} = 288$ and $QA = 10$.
  a. Find the circumference of $\odot Q$.
  b. Find the length of $\overset{\frown}{AC}$.
  c. Find the area of sector AQC.



Ex. 18

19. A point is randomly chosen on $\overline{AD}$. Find the probability that the point is on $\overline{AE}$.

20. A point is randomly chosen inside $\triangle ABC$. What is the probability that the point is inside:
  a. $\triangle ABD$? **b.** $\triangle BDE$?



Exs. 19, 20

Cumulative Review: Chapters 1–11

In Exercises 1–12 classify each statement as true or false.

A **1.** If point A lies on $\overrightarrow{BC}$, but not on $\overline{BC}$, then B is between A and C.

2. A true conditional always has a true converse.

3. The statement "If $ac = bc$, then $a = b$" is true for all real numbers a, b, and c.

4. If two parallel lines are cut by transversal t and t is perpendicular to one of the lines, then t must also be perpendicular to the other line.

5. If $\triangle ABC \cong \triangle DEF$ and $\angle A \cong \angle B$, then $\overline{DE} \cong \overline{EF}$.

6. If the opposite sides of a quadrilateral are congruent and the diagonals are perpendicular, then the quadrilateral must be a square.

7. If $\triangle GBS \sim \triangle JFK$, then $\frac{JF}{JK} = \frac{GB}{GS}$.

8. The length of the altitude to the hypotenuse of a right triangle is always the geometric mean between the lengths of the legs.

9. In any right triangle, the sine of one acute angle is equal to the cosine of the other acute angle.

10. If an angle inscribed in a circle intercepts a major arc, then the measure of the angle must be between 180 and 360.

11. The angle bisectors of an obtuse triangle intersect at a point that is equidistant from the three vertices.

12. If $JK = 10$, then the locus of points in space that are 4 units from J and 5 units from K is a circle.

13. Two lines that do not intersect are either _?_ or _?_.

14. In $\triangle RST$, $m\angle R = 2x + 10$, $m\angle S = 3x - 10$, and $m\angle T = 4x$.
  a. Find the numerical measure of each angle.
  b. Is $\triangle RST$ scalene, isosceles, or right? Why?

15. Use inductive thinking to guess the next number: 10, 9, 5, -4, -20, _?_.

16. If a diagonal of an equilateral quadrilateral is drawn, what method could be used to show that the two triangles formed are congruent?

17. A trapezoid has bases with lengths $x + 3$ and $3x - 1$ and a median of length 11. Find the value of x.

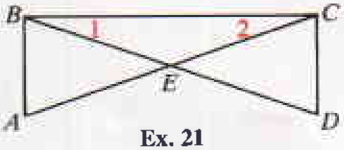
18. If 4, 7, and x are the lengths of the sides of a triangle and x is an integer, list the possible values for x.

19. Describe the locus of points in space that are 4 cm or less from a given point P.

20. Two similar rectangles have diagonals of $6\sqrt{3}$ and 9. Find the ratio of their perimeters and the ratio of their areas.

B **21.** Given: $\overline{AB} \perp \overline{BC}$; $\overline{DC} \perp \overline{BC}$; $\overline{AC} \cong \overline{BD}$
Prove: $\triangle BCE$ is isosceles.

22. Given: Quad. $EFGH$; $\overline{EF} \cong \overline{HG}$; $\overline{EF} \parallel \overline{HG}$
Prove: $\angle EHF \cong \angle GFH$



Ex. 21

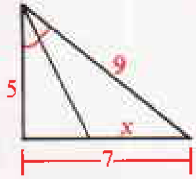
23. Use an indirect proof to show that no triangle has sides of length x, y, and $x + y$.

24. The legs of a right triangle are 4 cm and 8 cm long. What is the length of the median to the hypotenuse?

25. If a 45°-45°-90° triangle has legs of length $5\sqrt{2}$, find the length of the altitude to the hypotenuse.

26. The altitude to the hypotenuse of a 30°-60°-90° triangle divides the hypotenuse into segments with lengths in the ratio _?_:_?_.

27. Find the value of x in the diagram.



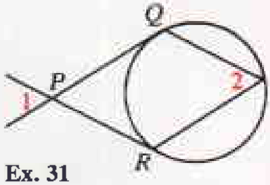
Ex. 27

28. In $\triangle DEF$, $m\angle F = 42$, $m\angle E = 90$, and $DE = 12$. Find EF to the nearest integer. (Use the table on page 311.)

29. In right $\triangle XYZ$ with hypotenuse $\overline{XZ}$ if $\cos X = \frac{7}{10}$ and $XZ = 24$, then to the nearest integer $XY =$ _?_.

30. If a tree is 20 m high and the distance from point P on the ground to the base of the tree is also 20 m, then the angle of elevation of the top of the tree from point P is _?_.

31. If $\overline{PQ}$ and $\overline{PR}$ are tangents to the circle and $m\angle 1 = 58$, find $m\angle 2$.



Ex. 31

32. $\triangle ABC$ is an isosceles right triangle with hypotenuse $\overline{AC}$ of length $2\sqrt{2}$. If medians $\overline{AD}$ and $\overline{BE}$ intersect at M, find AD and AM.

33. Draw two segments and let their lengths be x and y. Construct a segment of length t such that $t = \frac{2x^2}{y}$.

34. An equilateral triangle has perimeter 12 cm. Find its area.

35. Find the area of an isosceles trapezoid with legs 7 and bases 11 and 21.

36. a. Find the length of a 200° arc in a circle with diameter 24.
b. Find the area of the sector determined by this arc.

37. B and E are the respective midpoints of $\overline{AC}$ and $\overline{AD}$. Given that $AB = 9$, $BE = 6$, and $AE = 8$, find:
a. the perimeter of $\triangle ACD$
b. the ratio of the areas of $\triangle ABE$ and $\triangle ACD$