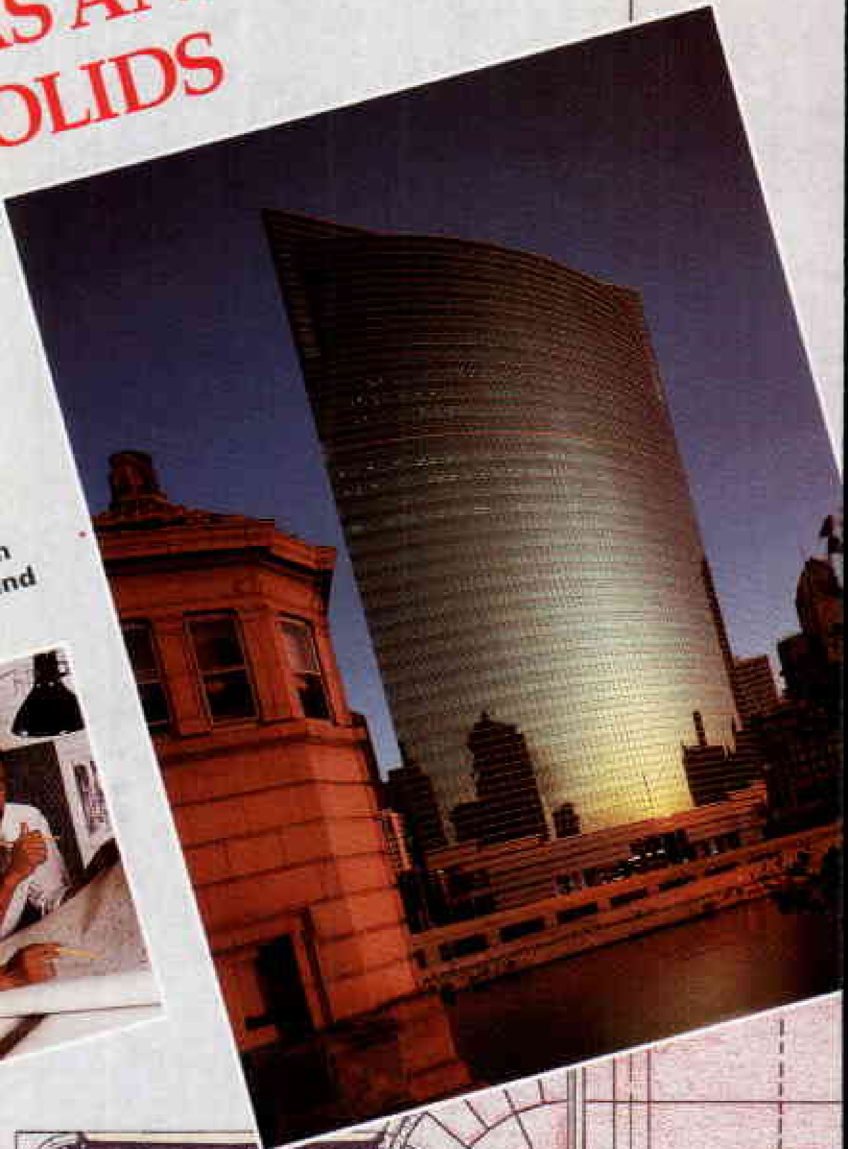
12 AREAS AND VOLUMES OF SOLIDS



The design of this unusual building reflects mastery of three-dimensional geometry and skill in combining curves and straight lines.

Important Solids

Objectives

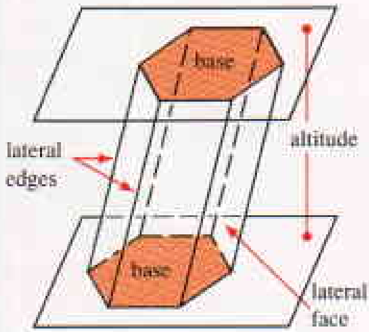
1. Identify the parts of prisms, pyramids, cylinders, and cones.
2. Find the lateral areas, total areas, and volumes of right prisms and regular pyramids.
3. Find the lateral areas, total areas, and volumes of right cylinders and right cones.

12-1 *Prisms*

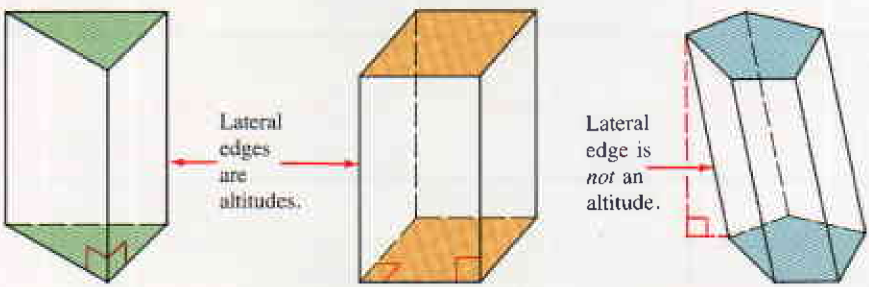
In this chapter you will be calculating surface areas and volumes of special solids. It is possible to begin with some postulates and then prove as theorems the formulas for areas and volumes of solids, as we did for plane figures. Instead, the formulas for solids will be stated as theorems, and informal arguments will be given to show you that the formulas are reasonable.

The first solid we will study is the **prism.** The two shaded faces of the prism shown are its **bases.** Notice that the bases are congruent polygons lying in parallel planes. An **altitude** of a prism is a segment joining the two base planes and perpendicular to both. The length of an altitude is the *height*, h, of the prism.

The faces of a prism that are not its bases are called **lateral faces.** Adjacent lateral faces intersect in parallel segments called **lateral edges.**



The lateral faces of a prism are parallelograms. If they are rectangles, the prism is a **right prism.** Otherwise the prism is an **oblique prism.** The diagrams below show that a prism is also classified by the shape of its bases. Note that in a right prism, the lateral edges are also altitudes.



Right triangular prism

Right rectangular prism
(Rectangular solid)

Oblique pentagonal prism

The surface area of a solid is measured in square units. The **lateral area** (L.A.) of a prism is the sum of the areas of its lateral faces. The **total area** (T.A.) is the sum of the areas of all its faces. Using B to denote the area of a base, we have the following formula.

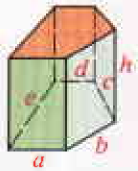
$$\text{T.A.} = \text{L.A.} + 2B$$

If a prism is a right prism, the next theorem gives us an easy way to find the lateral area.

Theorem 12-1

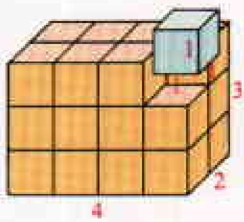
The lateral area of a right prism equals the perimeter of a base times the height of the prism. (L.A. = ph)

The formula for lateral area applies to any right prism. The right pentagonal prism can be used to illustrate the development of the formula:



$$\begin{aligned}\text{L.A.} &= ah + bh + ch + dh + eh \\ &= (a + b + c + d + e)h \\ &= \text{perimeter} \cdot h \\ &= ph\end{aligned}$$

Prisms have *volume* as well as area. A rectangular solid with square faces is a **cube.** Since each edge of the blue cube shown is 1 unit long, the cube is said to have a volume of 1 cubic unit. The larger rectangular solid has 3 layers of cubes, each layer containing $(4 \cdot 2)$ cubes. Hence its volume is $(4 \cdot 2) \cdot 3$, or 24 cubic units.



$$\begin{aligned}\text{Volume} &= \text{Base area} \times \text{height} \\ &= (4 \cdot 2) \cdot 3 \\ &= 24 \text{ cubic units}\end{aligned}$$

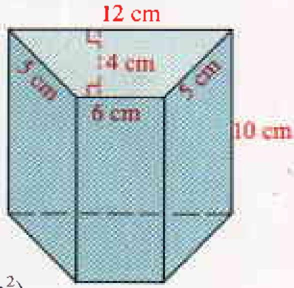
The same sort of reasoning is used to find the volume of any right prism.

Theorem 12-2

The volume of a right prism equals the area of a base times the height of the prism. ($V = Bh$)

Volume is measured in cubic units. Some common units for measuring volume are the cubic centimeter (cm^3) and the cubic meter (m^3).

Example 1 A right trapezoidal prism is shown. Find the (**a**) lateral area, (**b**) total area, and (**c**) volume.



Solution **a.** First find the perimeter of a base.
$p = 5 + 6 + 5 + 12 = 28$ (cm)

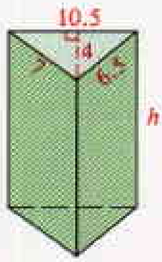
Now use the formula for lateral area.
$\text{L.A.} = ph = 28 \cdot 10 = 280\ (\text{cm}^2)$

b. First find the area of a base.
$B = \frac{1}{2} \cdot 4 \cdot (12 + 6) = 36\ (\text{cm}^2)$

Now use the formula for total area.
$\text{T.A.} = \text{L.A.} + 2B = 280 + 2 \cdot 36 = 352\ (\text{cm}^2)$

c. $V = Bh = 36 \cdot 10 = 360\ (\text{cm}^3)$

Example 2 A right triangular prism is shown. The volume is 315. Find the total area.



Solution First find the height of the prism.

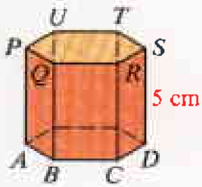
$$V = Bh$$
$$315 = (\tfrac{1}{2} \cdot 10.5 \cdot 4)h$$
$$315 = 21h$$
$$15 = h$$

Next find the lateral area.
$\text{L.A.} = ph = (10.5 + 6.5 + 7) \cdot 15 = 24 \cdot 15 = 360$

Now use the formula for total area.
$\text{T.A.} = \text{L.A.} + 2B = 360 + 2 \cdot 21 = 402$

Classroom Exercises

Exercises 1–6 refer to the right prism shown.



1. The prism is called a right __?__ prism.
2. How many lateral faces are there?
3. What kind of figure is each lateral face?
4. Name two lateral edges and an altitude.
5. The length of an altitude is called the __?__ of the prism.
6. Suppose the bases are regular hexagons with 4 cm edges.
   a. Find the lateral area. **b.** Find the base area.
   c. Find the total area. **d.** Find the volume.

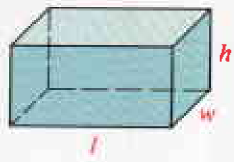
7. Can a prism have lateral faces that are triangles?
8. What is the minimum number of faces a prism can have?
9. If two prisms have equal volumes, must they also have equal total areas?
10. **a.** Since 1 yd = 3 ft, 1 yd^2 = __?__ ft^2 and 1 yd^3 = __?__ ft^3.
    b. Since 1 ft = __?__ in., 1 ft^2 = __?__ in.^2 and 1 ft^3 = __?__ in.^3
    c. Since 1 m = __?__ cm, 1 m^2 = __?__ cm^2 and 1 m^3 = __?__ cm^3.

Written Exercises

Exercises 1–6 refer to rectangular solids with dimensions l, w, and h. Complete the table.

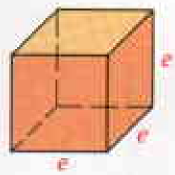
A

	1.	2.	3.	4.	5.	6.
l	6	50	6	?	9	$5x$
w	4	30	3	8	?	$4x$
h	2	15	?	5	2	$3x$
L.A.	?	?	?	?	60	?
T.A.	?	?	?	?	?	?
V	?	?	54	360	?	?



Exercises 7–12 refer to cubes with edges of length e. Complete the table.

	7.	8.	9.	10.	11.	12.
e	3	e	?	?	?	$2x$
T.A.	?	?	?	?	150	?
V	?	?	1000	64	?	?



13. Find the lateral area of a right pentagonal prism with height 13 and base edges 3.2, 5.8, 6.9, 4.7, and 9.4.

14. A right triangular prism has lateral area 120 cm^2. If the base edges are 4 cm, 5 cm, and 6 cm long, find the height of the prism.

15. If the edge of a cube is doubled, the total area is multiplied by _?_ and the volume is multiplied by _?_.

16. If the length, width, and height of a rectangular solid are all tripled, the lateral area is multiplied by _?_, the total area is multiplied by _?_, and the volume is multiplied by _?_.

Facts about the base of a right prism and the height of the prism are given. Sketch each prism and find its lateral area, total area, and volume.

17. Equilateral triangle with side 8; $h = 10$

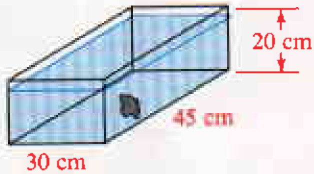
18. Triangle with sides 9, 12, 15; $h = 10$

B **19.** Isosceles triangle with sides 13, 13, 10; $h = 7$

20. Isosceles trapezoid with bases 10 and 4 and legs 5; $h = 20$

21. Rhombus with diagonals 6 and 8; $h = 9$

22. Regular hexagon with side 8; $h = 12$



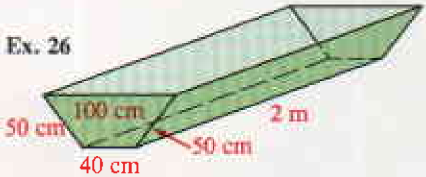
23. The container shown has the shape of a rectangular solid. When a rock is submerged, the water level rises 0.5 cm. Find the volume of the rock.

24. A driveway 30 m long and 5 m wide is to be paved with blacktop 3 cm thick. How much will the blacktop cost if it is sold at the price of \$175 per cubic meter?

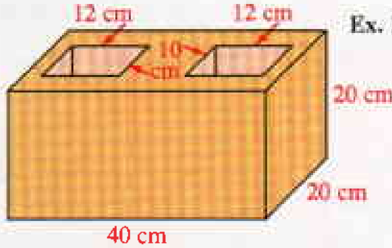
25. A brick with dimensions 20 cm, 10 cm, and 5 cm weighs 1.2 kg. A second brick of the same material has dimensions 25 cm, 15 cm, and 4 cm. What is its weight?

26. A drinking trough for horses is a right trapezoidal prism with dimensions shown below. If it is filled with water, about how much will the water weigh? (*Hint*: 1 m^3 of water weighs 1 metric ton.)

Ex. 26

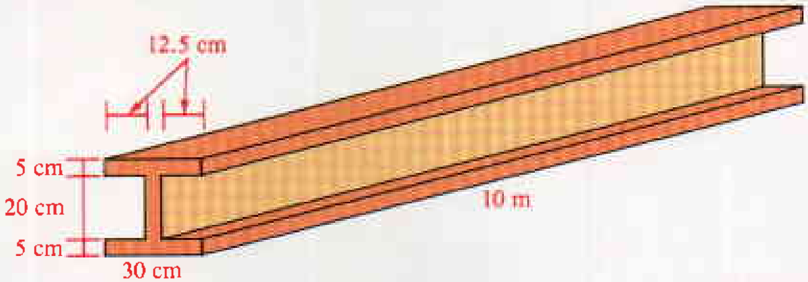


Ex. 27



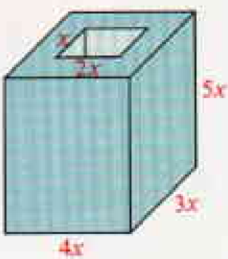
27. Find the weight, to the nearest kilogram, of the cement block shown. Cement weighs 1700 kg/m^3.

28. Find the weight, to the nearest 10 kg, of the steel I-beam shown below. Steel weighs 7860 kg/m^3.

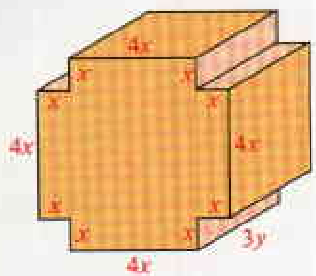


Find the volume and the total surface area of each solid in terms of the given variables.

29.

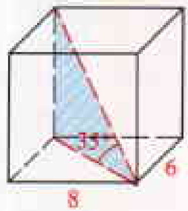


30.



31. The length of a rectangular solid is twice the width, and the height is three times the width. If the volume is 162 cm^3, find the total area of the solid.

32. A right prism has square bases with edges that are three times as long as the lateral edges. The prism's total area is 750 m^2. Find the volume.



33. A diagonal of a box forms a 35° angle with a diagonal of the base, as shown. Use trigonometry to approximate the volume of the box.

34. Refer to Exercise 33. Suppose another box has a base with dimensions 8 by 6 and a diagonal that forms a 70° angle with a diagonal of a base. Show that the ratio of the volumes of the two boxes is $\frac{\tan 35°}{\tan 70°}$.

C **35.** A right prism has height x and bases that are equilateral triangles with sides x. Show that the volume is $\frac{1}{4}x^3\sqrt{3}$.

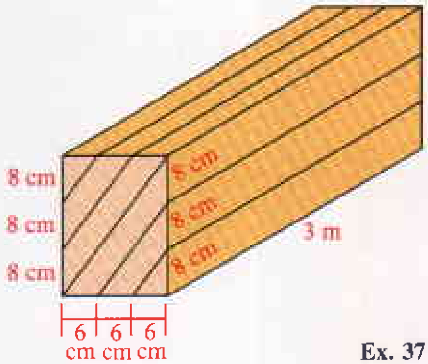
36. A right prism has height h and bases that are regular hexagons with sides s. Show that the volume is $\frac{3}{2}s^2h\sqrt{3}$.

37. A rectangular beam of wood 3 m long is cut into six pieces, as shown. Find the volume of each piece in cubic centimeters.

38. A diagonal of a cube joins two vertices not in the same face. If the diagonals are $4\sqrt{3}$ cm long, what is the volume?

39. All nine edges of a right triangular prism are congruent. Find the length of these edges if the volume is $54\sqrt{3}\text{ cm}^3$.

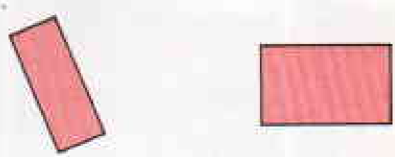
40. If the length and width of a rectangular solid are each decreased by 20%, by what percent must the height be increased for the volume to remain unchanged? Give your answer to the nearest whole percent.



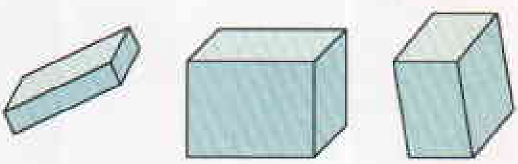
Ex. 37

Challenge

1. Given two rectangles, find one line that divides each rectangle into two parts of equal area.

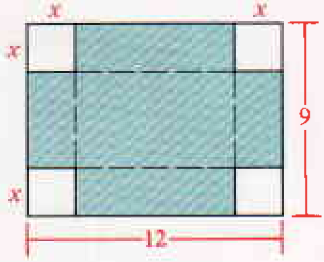


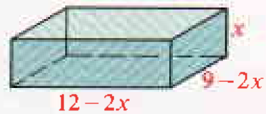
2. Given three rectangular solids, tell how to find one plane that divides each of these solids into two parts of equal volume.



♦ Computer Key-In

A manufacturing company produces metal boxes of different sizes by cutting out square corners from rectangular pieces of metal that measure 9 in. by 12 in. The metal is then folded along the dashed lines to form a box without a top. If a customer requests the box with the greatest possible volume, what dimensions should be used?





The volume, V, of the box can be expressed in terms of x.

$$\begin{aligned} V &= \text{length} \cdot \text{width} \cdot \text{height} \\ &= (12 - 2x) \cdot (9 - 2x) \cdot x \end{aligned}$$

To form a box, the possible values for x are $0 < x < \frac{9}{2}$.

The following computer program finds the volumes of the boxes produced for ten values of x from 0 to 4.5.

```
10  PRINT "X", "VOLUME"
15  PRINT
20  FOR X = 0 TO 4.5 STEP 0.5
30  LET V = (12 - 2 * X) * (9 - 2 * X) * X
40  PRINT X, V
50  NEXT X
60  END
```

X	VOLUME
0	0
.5	44
1	70
1.5	81
2	80
2.5	70
3	54
3.5	35
4	16
4.5	0

The print-out at the right shows that the maximum volume of the box probably occurs when the value of x is between 1 and 2. Also, the print-out shows that the maximum volume is about 81 in.^{3}

Exercises

1. To find a more accurate value for x, change line 20 to:

```
FOR X = 1 TO 2 STEP 0.1
```

Between what values of x does the maximum volume occur?

2. Modify line 20 to find the maximum volume, correct to the nearest tenth of a cubic inch. What are the length, width, and height, correct to the nearest tenth of an inch, of the box with maximum volume?

3. Suppose the manufacturing company cuts square corners out of pieces of metal that measure 8 in. by 15 in.

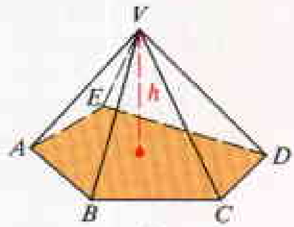
a. Express the volume in terms of x.

b. Find the maximum volume, correct to the nearest tenth of a cubic inch.

c. What are the length, width, and height of the box with maximum volume? Give each correct to the nearest tenth of an inch.

12-2 *Pyramids*

The diagram shows the pentagonal **pyramid** $V\text{-}ABCDE$. Point V is the **vertex** of the pyramid and pentagon $ABCDE$ is the **base.** The segment from the vertex perpendicular to the base is the **altitude** and its length is the *height*, h, of the pyramid.



The five triangular faces with V in common, such as $\triangle VAB$, are **lateral faces.** These faces intersect in segments called **lateral edges.**

Most of the pyramids you'll study will be **regular pyramids.** These are pyramids with the following properties:

(1) The base is a regular polygon.
(2) All lateral edges are congruent.
(3) All lateral faces are congruent isosceles triangles. The height of a lateral face is called **the slant height** of the pyramid. It is denoted by l.
(4) The altitude meets the base at its center, O.

Regular hexagonal pyramid

Example 1 A regular square pyramid has base edges 10 and lateral edges 13. Find **(a)** its slant height and **(b)** its height.

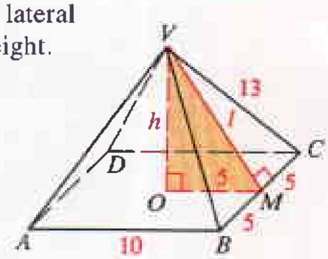
Solution Use the Pythagorean Theorem.

a. In rt. $\triangle VMC$,

$$l = \sqrt{13^2 - 5^2} = 12.$$

b. In rt. $\triangle VOM$,

$$h = \sqrt{12^2 - 5^2} = \sqrt{119}.$$

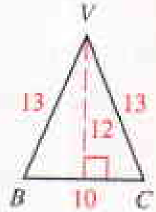


Example 2 Find the lateral area of the pyramid in Example 1.

Solution The four lateral faces are congruent.

area of $\triangle VBC = \frac{1}{2} \cdot 10 \cdot 12 = 60$

lateral area = area of 4 lateral faces
$= 4 \cdot$ area of $\triangle VBC$
$= 4 \cdot 60 = 240$



Example 2 illustrates a simple method for finding the lateral area of a regular pyramid. It is Method 1, summarized below.

To find the lateral area of a **regular** pyramid with n lateral faces:

Method 1 Find the area of one lateral face and multiply by n.

Method 2 Use the formula L.A. $= \frac{1}{2}pl$, stated as the next theorem.

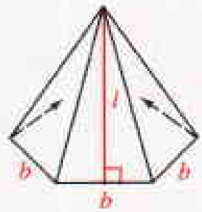
Theorem 12-3

The lateral area of a regular pyramid equals half the perimeter of the base times the slant height. (L.A. = $\frac{1}{2}pl$)

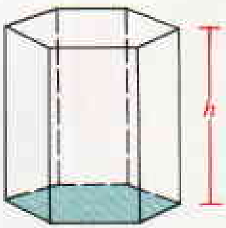
This formula is developed using Method 1 on the previous page. The area of one lateral face is $\frac{1}{2}bl$. Then:

$$\text{L.A.} = (\tfrac{1}{2}bl)n$$
$$= \tfrac{1}{2}(nb)l$$

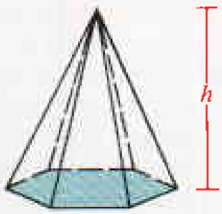
Since $nb = p$, $\quad$ L.A. $= \frac{1}{2}pl$



The prism and pyramid below have congruent bases and equal heights. Since the volume of the prism is Bh, the volume of the pyramid must be less than Bh. In fact, it is exactly $\frac{1}{3}Bh$. This result is stated as Theorem 12-4. Although no proof will be given, Classroom Exercise 1 and the Computer Key-In on pages 488–489 help justify the formula.



$V = Bh$



$V = \frac{1}{3}Bh$

Theorem 12-4

The volume of a pyramid equals one third the area of the base times the height of the pyramid. ($V = \frac{1}{3}Bh$)

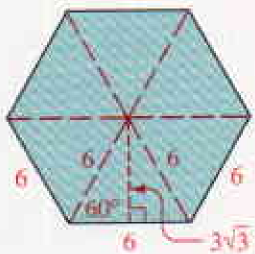
Example 3 Suppose the regular hexagonal pyramid shown at the right above Theorem 12-4 has base edges 6 and height 12. Find its volume.

Solution Find the area of the hexagonal base.

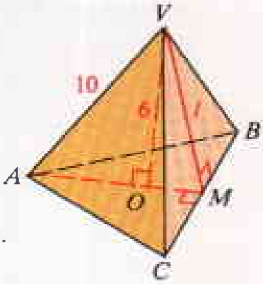
Divide the base into six equilateral triangles. Find the area of one triangle and multiply by 6.

Base area $= B = 6(\frac{1}{2} \cdot 6 \cdot 3\sqrt{3}) = 54\sqrt{3}$

Then $V = \frac{1}{3}Bh = \frac{1}{3} \cdot 54\sqrt{3} \cdot 12 = 216\sqrt{3}$



Example 4 A regular triangular pyramid has lateral edge 10 and height 6. Find the **(a)** lateral area and **(b)** volume.

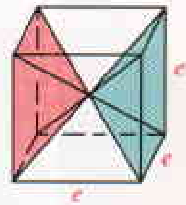


Solution

a. In rt. $\triangle VOA$, $AO = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$.
Since $AO = \frac{2}{3}AM$ (why?), $\frac{2}{3}AM = 8$,
$AM = 12$, and $OM = 4$.
$l = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$
In 30°-60°-90° $\triangle AMC$, $CM = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$.
Base edge $= BC = 2 \cdot 4\sqrt{3} = 8\sqrt{3}$
L.A. $= \frac{1}{2}pl = \frac{1}{2}(3 \cdot 8\sqrt{3}) \cdot 2\sqrt{13} = 24\sqrt{39}$

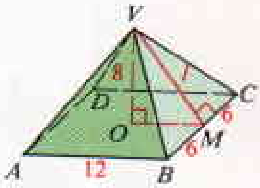
b. Area of base $= B = \frac{1}{2} \cdot BC \cdot AM = \frac{1}{2} \cdot 8\sqrt{3} \cdot 12 = 48\sqrt{3}$
$V = \frac{1}{3}Bh = \frac{1}{3} \cdot 48\sqrt{3} \cdot 6 = 96\sqrt{3}$

Classroom Exercises



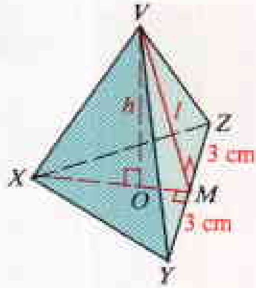
1. The diagonals of a cube intersect to divide the cube into six congruent pyramids as shown. The base of each pyramid is a face of the cube, and the height of each pyramid is $\frac{1}{2}e$.
   a. Use the formula for the volume of a cube to explain why the volume of each pyramid is $V = \frac{1}{6}e^3$.
   b. Use the formula in part (a) to show that $V = \frac{1}{3}Bh$. (*Note*: This exercise shows that $V = \frac{1}{3}Bh$ gives the correct result for *these* pyramids.)

V-ABCD is a regular square pyramid. Find numerical answers.



2. $OM =$ __?__
3. $l =$ __?__
4. Area of $\triangle VBC =$ __?__
5. L.A. $=$ __?__
6. Volume $=$ __?__
7. $VC =$ __?__

Each edge of pyramid V-XYZ is 6 cm. Find numerical answers.



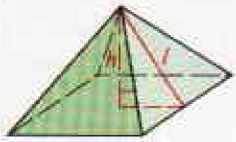
8. $XM =$ __?__
9. $XO =$ __?__
10. $h =$ __?__
11. Base area $=$ __?__
12. Volume $=$ __?__
13. Slant height $=$ __?__
14. L.A. $=$ __?__
15. T.A. $=$ __?__
16. Can the height of a regular pyramid be greater than the slant height? Explain.
17. Can the slant height of a regular pyramid be greater than the length of a lateral edge? Explain.
18. Can the area of the base of a regular pyramid be greater than the lateral area? Explain.

Written Exercises

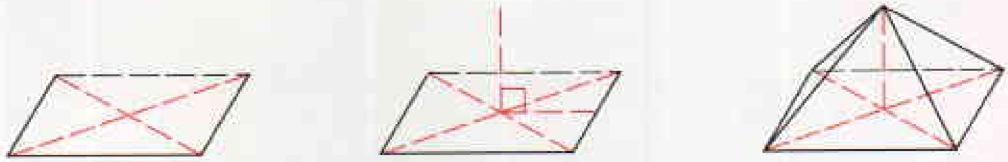
Copy and complete the table below for the regular square pyramid shown.

A

	1.	2.	3.	4.	5.	6.
height, *h*	4	12	24	?	?	6
slant height, *l*	5	13	?	12	5	?
base edge	?	?	14	?	8	?
lateral edge	?	?	?	15	?	10



You can use the following three steps to sketch a square pyramid.



(1) Draw a parallelogram for the base and sketch the diagonals.

(2) Draw a vertical segment at the point where the diagonals intersect.

(3) Join the vertex to the base vertices.

Sketch each pyramid, as shown above. Then find its lateral area.

7. A regular triangular pyramid with base edge 4 and slant height 6

8. A regular pentagonal pyramid with base edge 1.5 and slant height 9

9. A regular square pyramid with base edge 12 and lateral edge 10

10. A regular hexagonal pyramid with base edge 10 and lateral edge 13

For Exercises 11–14 sketch each square pyramid described. Then find its lateral area, total area, and volume.

11. base edge = 6, height = 4

12. base edge = 16, slant height = 10

13. height = 12, slant height = 13

14. base edge = 16, lateral edge = 17

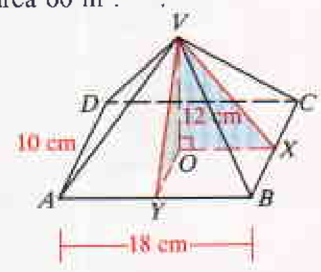
15. A pyramid has a base area of 16 cm^2 and a volume of 32 cm^3. Find its height.

16. A regular octagonal pyramid has base edge 3 m and lateral area 60 m^2. Find its slant height.

B **17.** *V-ABCD* is a pyramid with a rectangular base 18 cm long and 10 cm wide. *O* is the center of the rectangle. The height, *VO*, of the pyramid is 12 cm.

a. Find *VX* and *VY*.

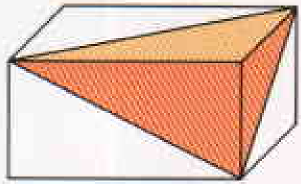
b. Find the lateral area of the pyramid. (Why can't you use the formula L.A. = $\frac{1}{2}pl$?)



18. Find the height and the volume of a regular hexagonal pyramid with lateral edges 10 ft and base edges 6 ft.

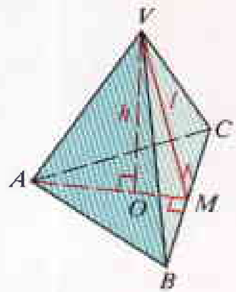
19. The shaded pyramid in the diagram is cut from a rectangular solid. How does the volume of the pyramid compare with the volume of the rectangular solid?

20. A pyramid and a prism both have height 8.2 cm and congruent hexagonal bases with area 22.3 cm^2. Give the ratio of their volumes. (*Hint*: You do *not* need to calculate their volumes.)



Ex. 19

Exercises 21–25 refer to the regular triangular pyramid shown below.



21. If $AM = 9$ and $VA = 10$, find h and l.

22. a. If $BC = 6$, find AM and AO.
b. If $BC = 6$ and $VA = 4$, find h and l.

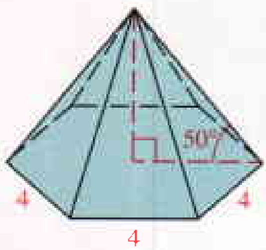
23. a. If $h = 4$ and $l = 5$, find OM, OA, and BC.
b. Find the lateral area and the volume.

24. If $VA = 5$ and $h = 3$, find the slant height, the lateral area, and the volume.

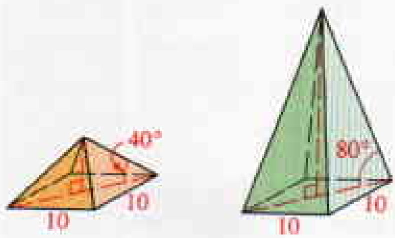
25. If $AB = 12$ and $VA = 10$, find the lateral area and the volume.

26. Find the volume of a regular hexagonal pyramid with height 8 cm and base edges 6 cm.

27. Use trigonometry to find the volume of the regular pyramid below to the nearest cubic unit.



Ex. 27



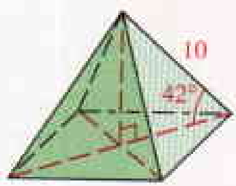
Ex. 28

28. Show that the ratio of the volumes of the two regular square pyramids shown above is $\frac{\tan 40°}{\tan 80°}$.

C **29.** All the edges of a regular triangular pyramid are x units long. Find the volume of the pyramid in terms of x.

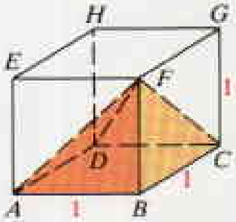
30. The base of a pyramid is a regular hexagon with sides y cm long. The lateral edges are $2y$ cm long. Find the volume of the pyramid in terms of y.

31. Use a calculator and trigonometry to find the volume of the regular square pyramid shown to the nearest cubic unit.

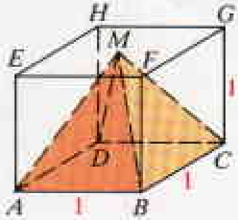


Ex. 31

★ **32.** Different pyramids are inscribed in two identical cubes, as shown below.
a. Which pyramid has the greater volume?
b. Which pyramid has the greater total area?

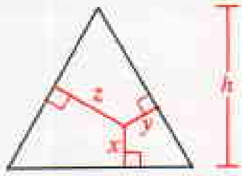


Pyramid *F-ABCD*



Pyramid *M-ABCD* has vertex *M* at the center of square *EFGH*.

Challenge



1. Accurately draw or construct a large equilateral triangle. Choose any point inside the triangle and carefully measure the distances x, y, z, and h. Then find $x + y + z$.

2. Now choose another point on or inside the triangle and find $x + y + z$. What do you notice? Why does this happen?

3. Use your answers in Exercises 1 and 2 to complete the following statement: From any point inside an equilateral triangle, the sum of the __?__ equals the __?__.

4. Generalize the statement in Exercise 3 from two dimensions to three dimensions.

Mixed Review Exercises

Copy and complete the table for circles.

	1.	2.	3.	4.	5.	6.	7.	8.
Radius	6	11	$\frac{1}{2}$	$3\sqrt{3}$	?	?	?	?
Circumference	?	?	?	?	10π	18π	?	?
Area	?	?	?	?	?	?	49π	15π

Draw a diagram for each exercise.

9. A circle is inscribed in a square with sides 24 mm. Find **(a)** the area of the circle and **(b)** the area of the square.

10. A square is inscribed in a circle with diameter $8\sqrt{2}$. Find **(a)** the perimeter of the square and **(b)** the circumference of the circle.

♦ Calculator Key-In

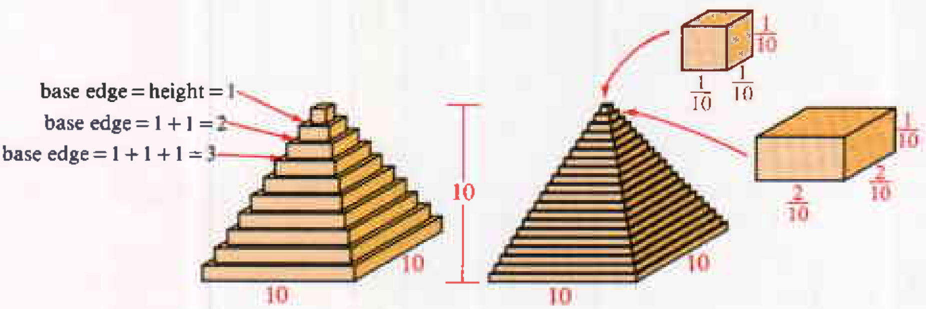
The Great Pyramid of King Cheops has a square base with sides 755 feet long. The original height was 481 feet, but the top part of the pyramid, which was 31 feet in height, has been destroyed. Approximately what percent of the original volume remains? Answer to the nearest hundredth of a percent.

♦ Computer Key-In

The earliest pyramids, which were built about 2750 B.C., are called *step pyramids* because the lateral faces are not triangles but a series of great stone steps. To find the volume of such a pyramid it is necessary to find the sum of the volumes of the steps, or layers. Each layer is a rectangular solid with a square base.

Let us consider a pyramid with base edges 10 and height 10. Suppose that this pyramid is made up of 10 steps with equal heights. The top layer is a cube (base edges equal the height), and the base edge for each succeeding layer increases by an amount equal to the height of a layer. As the left side of the diagram at the bottom of this page shows, the height of each step is $\frac{10}{10} = 1$, and the volume of the top layer is $V_1 = Bh = (1^2) \cdot 1 = 1$. The volumes of the second and third layers are $V_2 = (2^2) \cdot 1 = 4$ and $V_3 = (3^2) \cdot 1 = 9$, respectively. Continuing in this way, the total volume of the pyramid is:

$$V = 1^2 \cdot 1 + 2^2 \cdot 1 + 3^2 \cdot 1 + 4^2 \cdot 1 + \cdots + 9^2 \cdot 1 + 10^2 \cdot 1 = 385$$



Now consider another pyramid with the same base and height but having 100 steps instead of 10 steps. The height of each layer is $\frac{10}{100} = \frac{1}{10}$, and the volume of each layer is computed using the formula $V = Bh$:

$$\text{Volume of top layer} = \left(\frac{1}{10}\right)^2 \cdot \frac{1}{10}$$

$$\text{Volume of second layer} = \left(2 \cdot \frac{1}{10}\right)^2 \cdot \frac{1}{10} = \left(\frac{2}{10}\right)^2 \cdot \frac{1}{10}$$

$$\text{Volume of third layer} = \left(3 \cdot \frac{1}{10}\right)^2 \cdot \frac{1}{10} = \left(\frac{3}{10}\right)^2 \cdot \frac{1}{10}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\text{Volume of 99th layer} = \left(99 \cdot \frac{1}{10}\right)^2 \cdot \frac{1}{10} = \left(\frac{99}{10}\right)^2 \cdot \frac{1}{10}$$

$$\text{Volume of 100th layer} = \left(100 \cdot \frac{1}{10}\right)^2 \cdot \frac{1}{10} = \left(\frac{100}{10}\right)^2 \cdot \frac{1}{10}$$

Thus, the volume of the pyramid is:

$$V = \left(\frac{1}{10}\right)^2 \cdot \frac{1}{10} + \left(\frac{2}{10}\right)^2 \cdot \frac{1}{10} + \left(\frac{3}{10}\right)^2 \cdot \frac{1}{10} + \cdots + \left(\frac{99}{10}\right)^2 \cdot \frac{1}{10} + \left(\frac{100}{10}\right)^2 \cdot \frac{1}{10}$$

The following computer program finds the total volume for the given pyramid with N steps.

```
10  LET V = 0
20  PRINT "HOW MANY STEPS ARE THERE";
30  INPUT N
40  LET H = 10/N
50  FOR X = 1 TO N
60  LET V = V + (X * H) ↑2 * H
70  NEXT X
80  PRINT "VOLUME OF PYRAMID WITH ";N;" STEPS IS ";V
90  END
```

Exercises

1. RUN the given program to verify the volume of the 10-step pyramid and to find the volume of the 100-step pyramid.

2. **a.** Suppose that another pyramid with the same base and height has 1000 steps. RUN the program to find the volume.

b. Make a chart that shows the volume for the given number of steps: 10, 100, 500, 750, 900, 1000.

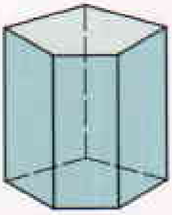
c. As the number of steps increases, what value do the volumes seem to be approaching?

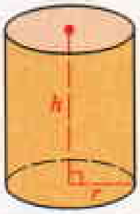
d. What is the volume of a regular square pyramid with base edge of 10 and height 10?

e. What can you conclude from comparing the answers to parts (b)–(d)?

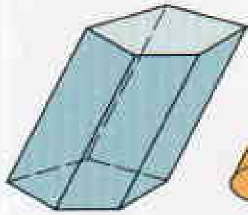
12-3 *Cylinders and Cones*

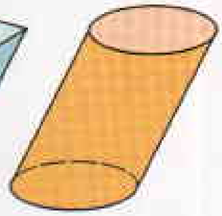
A **cylinder** is like a prism except that its bases are circles instead of polygons. In a **right cylinder,** the segment joining the centers of the circular bases is an **altitude.** The length of an altitude is called the *height*, h, of the cylinder. A radius of a base is also called a **radius,** r, of the cylinder.


Right prism



Right cylinder


Oblique prism


Oblique cylinder

The diagrams above show the relationship between prisms and cylinders. In the discussion and exercises that follow, the word "cylinder" will always refer to a right cylinder.

It is not surprising that the formulas for cylinders are related to those for prisms: L.A. $= ph$ and $V = Bh$. Since the base of a cylinder is a circle, we substitute $2\pi r$ for p and πr^2 for B and get the following formulas.

Theorem 12-5

The lateral area of a cylinder equals the circumference of a base times the height of the cylinder. (L.A. $= 2\pi rh$)

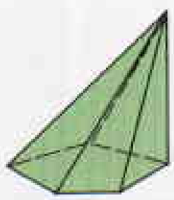
Theorem 12-6

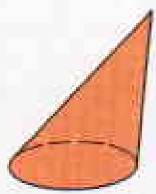
The volume of a cylinder equals the area of a base times the height of the cylinder. ($V = \pi r^2 h$)

A **cone** is like a pyramid except that its base is a circle instead of a polygon. The relationship between pyramids and cones is shown in the diagrams below.

Regular pyramid

Right cone


Oblique pyramid


Oblique cone

Note that "slant height" applies only to a regular pyramid and a right cone. We will use the word "cone" to refer to a right cone.

The formulas for cones are related to those for pyramids: L.A. $= \frac{1}{2}pl$ and $V = \frac{1}{3}Bh$. Since the base of a cone is a circle, we again substitute $2\pi r$ for p and πr^2 for B and get the following formulas.

Theorem 12-7

The lateral area of a cone equals half the circumference of the base times the slant height. (L.A. $= \frac{1}{2} \cdot 2\pi r \cdot l$, or L.A. $= \pi rl$)

Theorem 12-8

The volume of a cone equals one third the area of the base times the height of the cone. ($V = \frac{1}{3}\pi r^2 h$)

So far our study of solids has not included formulas for oblique solids. The volume formulas for cylinders and cones, but *not* the area formulas, can be used for the corresponding oblique solids. (See the Extra on pages 516–517.)

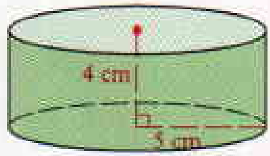
Example 1 A cylinder has radius 5 cm and height 4 cm. Find the **(a)** lateral area, **(b)** total area, and **(c)** volume of the cylinder.

Solution

a. L.A. $= 2\pi rh = 2\pi \cdot 5 \cdot 4 = 40\pi$ (cm^2)

b. T.A. $=$ L.A. $+ 2B$
$= 40\pi + 2(\pi \cdot 5^2) = 90\pi$ (cm^2)

c. $V = \pi r^2 h = \pi \cdot 5^2 \cdot 4 = 100\pi$ (cm^3)



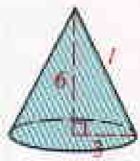
Example 2 Find the **(a)** lateral area, **(b)** total area, and **(c)** volume of the cone shown.

Solution

a. First use the Pythagorean Theorem to find l.
$l = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$
L.A. $= \pi rl = \pi \cdot 3 \cdot 3\sqrt{5} = 9\pi\sqrt{5}$

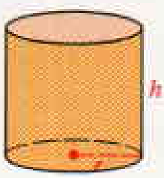
b. T.A. $=$ L.A. $+ B = 9\pi\sqrt{5} + \pi \cdot 3^2 = 9\pi\sqrt{5} + 9\pi$

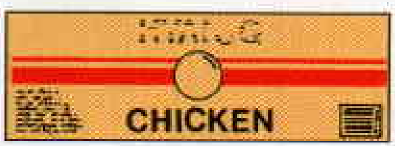
c. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 3^2 \cdot 6 = 18\pi$



Classroom Exercises

1. **a.** When the label of a soup can is cut off and laid flat, it is a rectangular piece of paper. (See the diagram below.) How are the length and width of this rectangle related to r and h?
   b. What is the area of this rectangle?

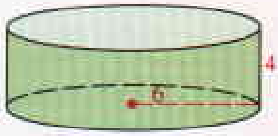




2. a. Find the lateral areas of cylinders I, II, and III.
   b. Notice that the height of II is twice the height of I. Is the lateral area of II twice the lateral area of I?
   c. Notice that the radius of III is twice the radius of I. Is the lateral area of III twice the lateral area of I?
3. a. Find the total areas of cylinders I, II, and III.
   b. Are the ratios of the total areas the same as those of the lateral areas in Exercise 2?
4. a. Find the volumes of cylinders I, II, and III.
   b. Notice that the height of II is twice the height of I. Is the volume of II twice the volume of I?
   c. Notice that the radius of III is twice the radius of I. Is the volume of III twice the volume of I?

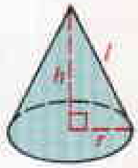
I


II


III


Complete the table for the cone shown.

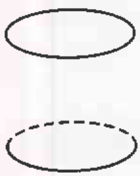
	r	h	l	L.A.	T.A.	V
5.	3	4	?	?	?	?
6.	?	12	13	?	?	?
7.	6 cm	?	10 cm	?	?	?



8. Describe the intersection of a plane and a cone if the plane is the perpendicular bisector of the altitude of the cone.

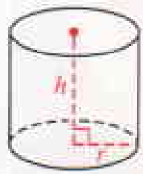
Written Exercises

You can use the following three steps to sketch a cylinder.



(1) Draw two congruent ovals, one above the other.

(2) Join the ovals with two vertical segments.



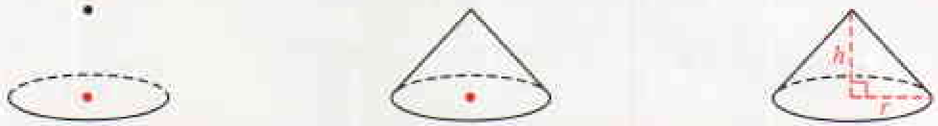
(3) Draw in the altitude and a radius.

Sketch each cylinder. Then find its lateral area, total area, and volume.

A 1. $r = 4$; $h = 5$ 2. $r = 8$; $h = 10$ 3. $r = 4$; $h = 3$ 4. $r = 8$; $h = 15$

5. The volume of a cylinder is 64π. If $r = h$, find r.
6. The lateral area of a cylinder is 18π. If $h = 6$, find r.
7. The volume of a cylinder is 72π. If $h = 8$, find the lateral area.
8. The total area of a cylinder is 100π. If $r = h$, find r.

You can use the following three steps to sketch a cone.



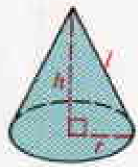
(1) Draw an oval and a vertex point over the center of the oval.

(2) Join the vertex to the oval, as shown.

(3) Draw in the altitude and a radius.

Sketch each cone. Copy and complete the table.

	r	h	l	L.A.	T.A.	V
9.	4	3	?	?	?	?
10.	8	6	?	?	?	?
11.	12	?	13	?	?	?
12.	?	2	6	?	?	?
13.	?	?	15	180π	?	?
14.	21	?	?	609π	?	?
15.	15	?	?	?	?	600π
16.	9	?	?	?	?	324π



17. In Exercises 9 and 10, the ratio of the radii is $\frac{4}{8}$, or $\frac{1}{2}$, and the ratio of the heights is $\frac{3}{6}$, or $\frac{1}{2}$. Use the answers you found for these two exercises to determine the ratios of the following:
a. lateral areas **b.** total areas **c.** volumes

18. A manufacturer needs to decide which container to use for packaging a product. One container is twice as wide as another but only half as tall. Which container holds more, or do they hold the same amount? Guess first and then calculate the ratio of their volumes.

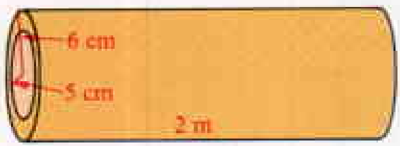
19. A cone and a cylinder both have height 48 and radius 15. Give the ratio of their volumes without calculating the two volumes.

B **20.** **a.** Guess which contains more, the can or the bottle. (Assume that the top part of the bottle is a complete cone.)
b. See if your guess is right by finding the volumes of both.

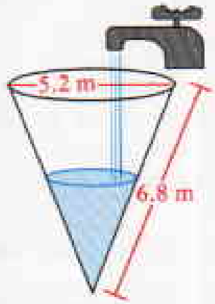
21. A solid metal cylinder with radius 6 cm and height 18 cm is melted down and recast as a solid cone with radius 9 cm. Find the height of the cone.

Ex. 20

22. A pipe is 2 m long and has inside radius 5 cm and outside radius 6 cm. Find the volume of metal contained in the pipe to the nearest cubic centimeter. Use $\pi \approx 3.14$.



Ex. 22



Ex. 23

23. Water is pouring into a conical (cone-shaped) reservoir at the rate of 1.8 m^3 per minute. Find, to the nearest minute, the number of minutes it will take to fill the reservoir. Use $\pi \approx 3.14$.

24. Two water pipes of the same length have inside diameters of 6 cm and 8 cm. These two pipes are replaced by a single pipe of the same length, which has the same capacity as the smaller pipes combined. What is the inside diameter of the new pipe?

25. The total area of a cylinder is 40π. If $h = 8$, find r.

26. The total area of a cylinder is 90π. If $h = 12$, find r.

27. In rectangle $ABCD$, $AB = 10$ and $AD = 6$.

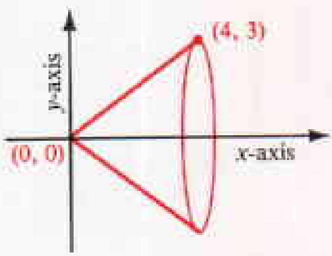
a. The rectangle is rotated in space about $\overline{AB}$. Describe the solid that is formed and find its volume.

b. Answer part (a) if the rectangle is rotated about $\overline{AD}$.

28. **a.** The segment joining (0, 0) and (4, 3) is rotated about the x-axis, forming the lateral surface of a cone. Find the lateral area and the volume of this cone.

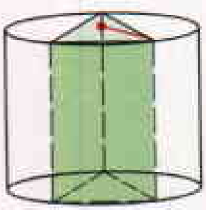
b. Sketch the cone that would be formed if the segment had been rotated about the y-axis. Find the lateral area and the volume of this cone.

c. Are your answers to parts (a) and (b) the same?



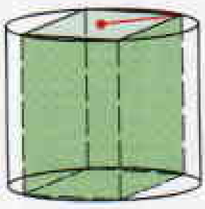
29. Each prism shown below is inscribed in a cylinder with height 10 and radius 6. Find the volume and lateral area of each prism.

a.



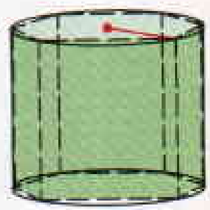
Base is an equilateral triangle.

b.



Base is a square.

c.



Base is a regular hexagon.

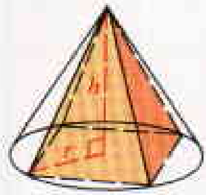
30. An equilateral triangle with 6 cm sides is rotated about an altitude. Draw a diagram and find the volume of the solid formed.

31. A square is rotated in space about a side s. Describe the solid formed and find its volume in terms of s.

32. A cylinder with height 10 and radius 6 is inscribed in a square prism. Make a sketch. Then find the volume of the prism.

33. A regular square pyramid with base edge 4 cm is inscribed in a cone with height 6 cm. What is the volume of the cone?

34. A regular square pyramid is inscribed in a cone with radius 4 cm and height 4 cm.
a. What is the volume of the pyramid?
b. Find the slant heights of the cone and the pyramid.



Exs. 33, 34

35. A cone is inscribed in a regular square pyramid with slant height 9 cm and base edge 6 cm. Make a sketch. Then find the volume of the cone.

36. The lateral area of a cone is three-fifths the total area. Find the ratio of the radius and the slant height.

C **37.** A regular hexagonal pyramid with base edge 6 and height 8 is inscribed in a cone. Find the lateral areas of the cone and the pyramid.

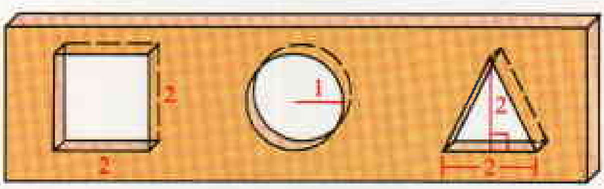
38. A 120° sector is cut out of a circular piece of tin with radius 6 in. and bent to form the lateral surface of a cone. What is the volume of the cone?

39. In $\triangle ABC$, $AB = 15$, $AC = 20$, and $BC = 25$. The triangle is rotated in space about $\overline{BC}$. Find the volume of the solid formed.

40. An equilateral triangle with sides of length s is rotated in space about one side. Show that the volume of the solid formed is $\frac{1}{4}\pi s^3$.

Challenge

A piece of wood contains a square hole, a circular hole, and a triangular hole, as shown. Explain how one block of wood in the shape of a cube with 2 cm edges can be cut down so that it can pass through, and exactly fit, all three holes.



Self-Test 1

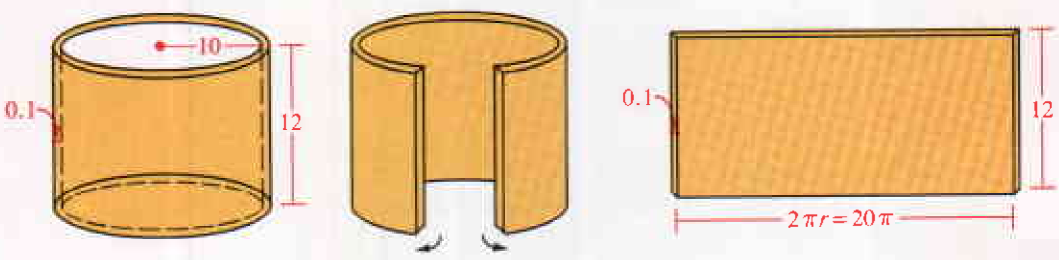
For Exercises 1–5 find the lateral area, total area, and volume of each solid.

1. A rectangular solid with length 10, width 8, and height 4.5
2. A regular square pyramid with base edge 24 and slant height 13
3. A cylinder with radius 10 in. and height 7 in.
4. A right hexagonal prism with height 5 cm and base edge 6 cm
5. A cone with height 12 and radius 9

6. The total area of a cube is 2400 m^2. Find the volume.
7. A solid metal cylinder with radius 2 and height 2 is recast as a solid cone with radius 2. Find the height of the cone.
8. A prism with height 2 m and a pyramid with height 5 m have congruent triangular bases. Find the ratio of their volumes.

♦ Calculator Key-In

1. A cylinder has radius 10 and height 12. Suppose that the lateral surface of the cylinder is covered with a thin coat of paint having thickness 0.1. The volume of the paint can be calculated approximately or exactly.
   a. Use the diagrams below to explain the following formula.
   Approximate volume = (lateral area of cylinder) · (thickness of paint)
   $$V \approx (2\pi rh) \cdot (t)$$



   b. Why is this formula only an approximation of the volume?
2. Use a calculator and the formula to find the approximate volume of paint for each thickness: 0.1, 0.01, 0.001.
3. The exact volume of paint can be found by subtracting the volume of the inner cylinder (the given cylinder) from the volume of the outer cylinder (the given cylinder plus paint). Use a calculator to evaluate the exact volume of paint for each thickness: 0.1, 0.01, 0.001.
4. Compare the values for the approximate volume and exact volume for each thickness in Exercises 2 and 3. What can you conclude?

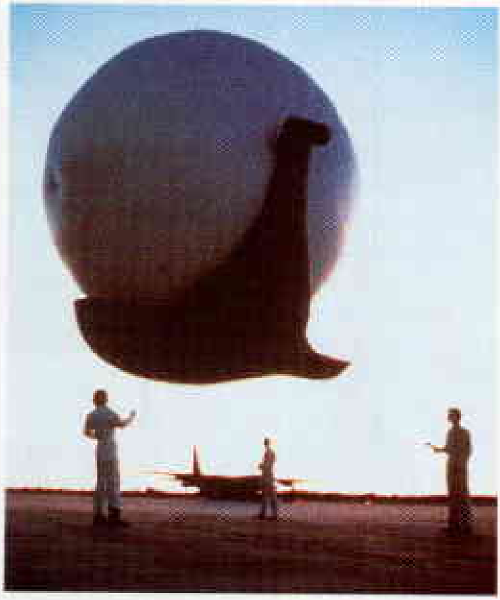
Similar Solids

Objectives

1. Find the area and the volume of a sphere.
2. State and apply the properties of similar solids.

12-4 *Spheres*

Recall (page 329) that a sphere is the set of all points that are a given distance from a given point. The sphere has many useful applications. One recent application is the development of a spherical blimp. An experimental model of the blimp is shown in the photograph. A spherical shape was selected for this blimp because a sphere gives excellent mobility, stability, hovering capabilities, and lift. The rotation of the top of a sphere away from the direction in which the sphere is traveling provides lifting power.



The surface area and the volume of a sphere are given by the formulas below. After some examples showing how these formulas are used, justifications of the formulas will be presented.

Theorem 12-9

The area of a sphere equals 4π times the square of the radius. ($A = 4\pi r^2$)

Theorem 12-10

The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius. ($V = \frac{4}{3}\pi r^3$)

Example 1 Find the area and the volume of a sphere with radius 2 cm.

Solution

$$A = 4\pi r^2 = 4\pi \cdot 2^2 = 16\pi \text{ (cm}^2\text{)}$$

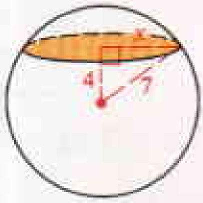
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 2^3 = \frac{32\pi}{3} \text{ (cm}^3\text{)}$$

Example 2 The area of a sphere is 256π. Find the volume.

Solution To find the volume, first find the radius.

(1) $A = 256\pi = 4\pi r^2$
$64 = r^2$
$8 = r$

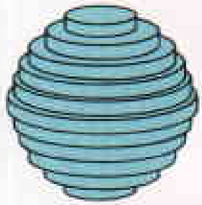
(2) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 8^3$
$= \frac{2048\pi}{3}$



Example 3 A plane passes 4 cm from the center of a sphere with radius 7 cm. Find the area of the circle of intersection.

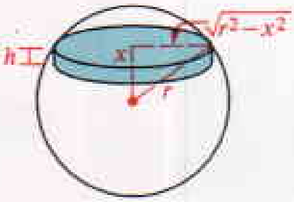
Solution Let x = radius of the circle.
$x = \sqrt{7^2 - 4^2} = \sqrt{33}$
Area $= \pi x^2 = \pi(\sqrt{33})^2 = 33\pi$ (cm^2)

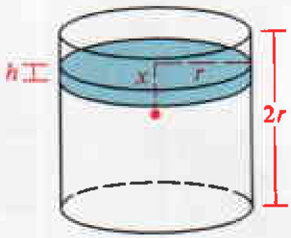
Justification of the Volume Formula (Optional)

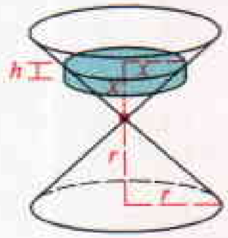


Any solid can be approximated by a stack of thin circular discs of equal thickness, as shown by the sphere drawn at the right. Each disc is actually a cylinder with height h.

The sphere, the cylinder, and the double cone below all have radius r and height $2r$. Look at the disc that is x units above the center of each solid.





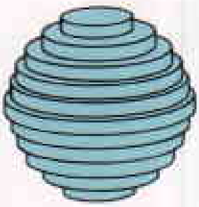


Disc volume:
$\pi(\sqrt{r^2 - x^2})^2 h = \pi(r^2 - x^2)h$
$= \pi r^2 h - \pi x^2 h$

Disc volume: $\pi r^2 h$

Disc volume: $\pi x^2 h$

Note from the calculations above that no matter what x is, the volume of the first disc equals the difference between the volumes of the other two discs.



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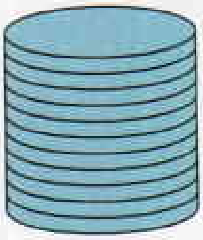

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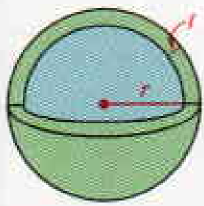
Total volume of discs in sphere = Total volume of discs in cylinder − Total volume of discs in double cone

The relationship on the bottom of the previous page holds if there are just a few discs approximating each solid or very many discs. If there are very many discs, their total volume will be practically the same as the volume of the solid. Thus:

$$\begin{aligned}\text{Volume of sphere} &= \text{volume of cylinder} - \text{volume of double cone}\\ &= \pi r^2 \cdot 2r - 2(\tfrac{1}{3}\pi r^2 \cdot r)\\ &= 2\pi r^3 - \tfrac{2}{3}\pi r^3\\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

Justification of the Area Formula (Optional)

Imagine a rubber ball with inner radius r and rubber thickness t. To find the volume of the rubber, we can use the formula for the volume of a sphere. We just subtract the volume of the inner sphere from the volume of the outer sphere.



$$\begin{aligned}\text{Exact volume of rubber} &= \tfrac{4}{3}\pi(r + t)^3 - \tfrac{4}{3}\pi r^3\\ &= \tfrac{4}{3}\pi[(r + t)^3 - r^3]\\ &= \tfrac{4}{3}\pi[r^3 + 3r^2t + 3rt^2 + t^3 - r^3]\\ &= 4\pi r^2 t + 4\pi rt^2 + \tfrac{4}{3}\pi t^3\end{aligned}$$

The volume of the rubber can be found in another way as well. If we think of a small piece of the rubber ball, its approximate volume would be its outer area A times its thickness t. The same thing is true for the whole ball.

$$\begin{aligned}\text{Volume of rubber} &\approx \text{Surface area} \cdot \text{thickness}\\ V &\approx A \cdot t\end{aligned}$$

Now we can equate the two formulas for the volume of the rubber:

$$A \cdot t \approx 4\pi r^2 t + 4\pi rt^2 + \tfrac{4}{3}\pi t^3$$

If we divide both sides of the equation by t, we get the following result:

$$A \approx 4\pi r^2 + 4\pi rt + \tfrac{4}{3}\pi t^2$$

This approximation for A gets better and better as the layer of rubber gets thinner and thinner. As t approaches zero, the last two terms in the formula for A also approach zero. As a result, the surface area gets closer and closer to $4\pi r^2$. Thus:

$$A = 4\pi r^2$$

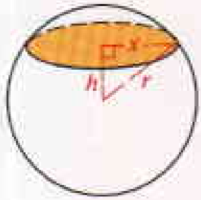
This is exactly what we would expect, since the surface area of a ball clearly depends on the size of the radius, not on the thickness of the rubber.

Classroom Exercises

Copy and complete the table for spheres.

	1.	2.	3.	4.	5.	6.
Radius	1	8	$3t$	?	?	?
Area	?	?	?	36π	100π	?
Volume	?	?	?	?	?	$\frac{4000\pi}{3}$

A plane passes h cm from the center of a sphere with radius r cm. Find the area of the circle of intersection, shaded in the diagram, for the given values.



7. $r = 5$, $h = 3$

8. $r = 17$, $h = 8$

9. $r = 7$, $h = 6$

Written Exercises

Copy and complete the table for spheres.

A

	1.	2.	3.	4.	5.	6.	7.	8.
Radius	7	5	$\frac{1}{2}$	$\frac{3}{4}k$	?	?	$\sqrt{2}$	?
Area	?	?	?	?	64π	324π	?	?
Volume	?	?	?	?	?	?	?	288π

9. If the radius of a sphere is doubled, the area of the sphere is multiplied by __?__ and the volume is multiplied by __?__.

10. Repeat Exercise 9 if the radius of the sphere is tripled.

11. The area of a sphere is π cm^2. Find the diameter of the sphere.

12. The volume of a sphere is 36π m^3. Find its area.

13. Find the area of the circle formed when a plane passes 2 cm from the center of a sphere with radius 5 cm.

14. Find the area of the circle formed when a plane passes 7 cm from the center of a sphere with radius 8 cm.

15. A sphere has radius 2 and a hemisphere (''half'' a sphere) has radius 4. Compare their volumes.

16. A scoop of ice cream with diameter 6 cm is placed in an ice-cream cone with diameter 5 cm and height 12 cm. Is the cone big enough to hold all the ice cream if it melts?

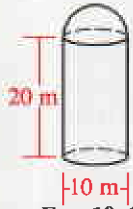
Ex. 16

17. Approximately 70% of the Earth's surface is covered by water. Use a calculator to find the area covered by water to the nearest million square kilometers. (The radius of the Earth is approximately 6380 km.)

18. **a.** Find the volume, correct to the nearest cubic centimeter, of a sphere inscribed in a cube with edges 6 cm long. Use $\pi \approx 3.14$.
b. Find the volume of the region inside the cube but outside the sphere.

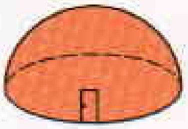
B **19.** A silo of a barn consists of a cylinder capped by a hemisphere, as shown. Find the volume of the silo.

20. About two cans of paint are needed to cover the hemispherical dome of the silo shown. Approximately how many cans are needed to paint the rest of the silo's exterior?



Exs. 19, 20

21. An experimental one-room house is a hemisphere with a floor. If three cans of paint are needed to cover the floor, how many cans will be needed to paint the ceiling? (Ignore door and windows.)



Ex. 21

22. A hemispheric bowl with radius 25 contains water whose depth is 10. What is the area of the water's surface?

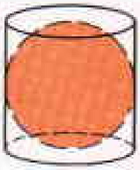
23. A solid metal ball with radius 8 cm is melted down and recast as a solid cone with the same radius.
a. What is the height of the cone?
b. Use a calculator to show that the lateral area of the cone is about 3% more than the area of the sphere.

24. Four solid metal balls fit snugly inside a cylindrical can. A geometry student claims that two extra balls of the same size can be put into the can, provided all six balls can be melted and the molten liquid poured into the can. Is the student correct? (*Hint:* Let the radius of each ball be r.)

Ex. 24

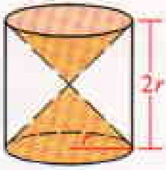
25. A sphere with radius r is inscribed in a cylinder. Find the volume of the cylinder in terms of r.

26. A sphere is inscribed in a cylinder. Show that the area of the sphere equals the lateral area of the cylinder.



Exs. 25, 26

27. A double cone is inscribed in the cylinder shown. Find the volume of the space inside the cylinder but outside the double cone.

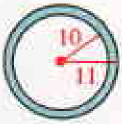


Ex. 27

28. A hollow rubber ball has outer radius 11 cm and inner radius 10 cm.
a. Find the exact volume of the rubber. Then evaluate the volume to the nearest cubic centimeter.
b. The volume of the rubber can be approximated by the formula:

$$V \approx \text{inner surface area} \cdot \text{thickness of rubber}$$

Use this formula to approximate V. Compare your answer with the answer in part (a).
c. Is the approximation method used in part (b) better for a ball with a thick layer of rubber or a ball with a thin layer?



Ex. 28

29. A circle with diameter 9 in. is rotated about a diameter. Find the area and volume of the solid formed.

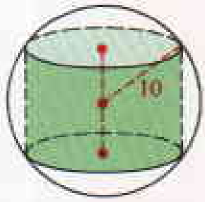
30. A cylinder with height 12 is inscribed in a sphere with radius 10. Find the volume of the cylinder.

C **31.** A cylinder with height $2x$ is inscribed in a sphere with radius 10.

a. Show that the volume of the cylinder, V, is
$$2\pi x(100 - x^2).$$

b. By using calculus, one can show that V is maximum when $x = \frac{10\sqrt{3}}{3}$. Substitute this value for x to find the maximum volume V.

c. (Optional) Use a calculator or a computer to evaluate $V = 2\pi x(100 - x^2)$ for various values of x between 0 and 10. Show that the maximum volume V occurs when $x \approx 5.77$.



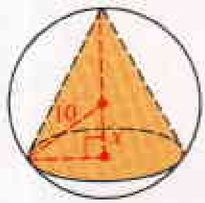
Exs. 30, 31

32. A cone is inscribed in a sphere with radius 10, as shown.

a. Show that the volume of the cone, V, is
$$\tfrac{1}{3}\pi(100 - x^2)(10 + x).$$

b. By using calculus, one can show that V is maximum when $x = \frac{10}{3}$. Substitute this value for x to find the maximum volume V.

c. (Optional) Use a calculator or a computer to evaluate the volume for various values of x between 0 and 10. Show that the maximum volume V occurs when $x = \frac{10}{3}$.

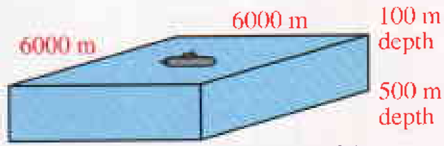


33. Sketch two intersecting spheres with radii 15 cm and 20 cm, respectively. The centers of the spheres are 25 cm apart. Find the area of the circle that is formed by the intersection. (*Hint*: Use Exercise 42 on page 289.)

34. A sphere is inscribed in a cone with radius 6 cm and height 8 cm. Find the volume of the sphere.

Challenge

In a training exercise, two research submarines are assigned to take any two positions in the ocean at a depth between 100 m and 500 m and within a square area with 6000 m sides. If one submarine takes its position in the center of the square area at the 100-meter depth, what is the probability that the other submarine is within 400 m? Use $\pi \approx 3.14$. (*Hint*: Consider each submarine to be a point in a rectangular solid. Use the ideas of geometric probability.)

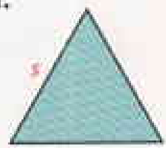


(Not to scale)

♦ Calculator Key-In

1. The purpose of this exercise is to suggest the following statement: Of all figures in a plane with a fixed perimeter, the circle has the greatest possible area. If each regular polygon below has perimeter 60 mm and the circle has circumference 60 mm, find the area of each to the nearest square millimeter.

a.



$s = \underline{\ ?\ }$

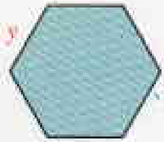
$A \approx \underline{\ ?\ }$

b.

$x = \underline{\ ?\ }$

$A = \underline{\ ?\ }$

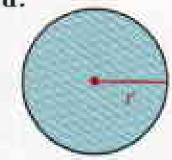
c.



$y = \underline{\ ?\ }$

$A \approx \underline{\ ?\ }$

d.

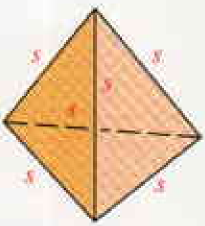


$r \approx \underline{\ ?\ }$

$A \approx \underline{\ ?\ }$

2. The regular pyramid, the cube, and the sphere below all have total surface area 600 mm^3. Find the volume of each to the nearest cubic millimeter.

a.



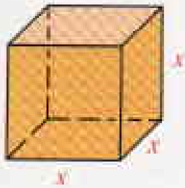
$\text{T.A.} = 4\left(\frac{s^2\sqrt{3}}{4}\right) = 600\ \text{mm}^2$

$s \approx \underline{\ ?\ }$

$V = \frac{s^3\sqrt{2}}{12} \approx \underline{\ ?\ }$

(See Ex. 29, page 486.)

b.

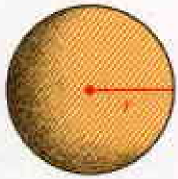


$\text{T.A.} = 6x^2 = 600\ \text{mm}^2$

$x = \underline{\ ?\ }$

$V = x^3 = \underline{\ ?\ }$

c.



$\text{T.A.} = 4\pi r^2 = 600\ \text{mm}^2$

$r \approx \underline{\ ?\ }$

$V = \frac{4}{3}\pi r^3 \approx \underline{\ ?\ }$

d. Use the results of parts (a)–(c) to complete the following statement, which is similar to the one in Exercise 1: Of all solid figures with a fixed $\underline{\ ?\ }$, the $\underline{\ ?\ }$ has the $\underline{\ ?\ }$.

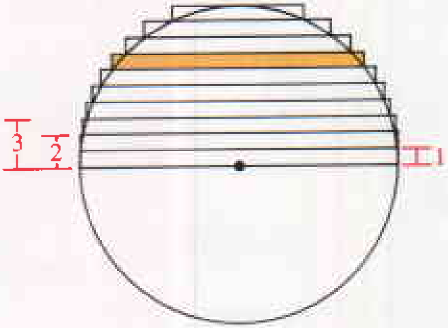
★ **3.** Suppose the plane figures in Exercise 1 all have area 900 cm^2. Find the perimeter of each polygon and the circumference of the circle to the nearest centimeter. What do your answers suggest about all plane figures with a fixed area?

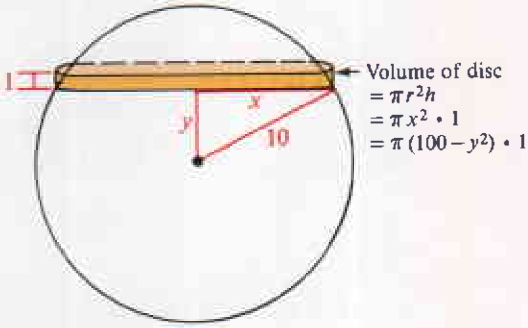
★ **4.** Suppose the solid figures in Exercise 2 all have volume 1000 cm^3. Find the total surface area of each to the nearest square centimeter. What do your answers suggest about all solid figures with a fixed volume?

◆ Computer Key-In

The volume of a sphere with radius 10 can be approximated by cylindrical discs with equal heights, as discussed on page 498. It is convenient to work with the upper half of the sphere, then double the result.

Suppose you use ten discs to approximate the upper hemisphere, as shown at the left below.





The diagram at the right above shows that the volume of a disc y units from the center of the sphere is $V = \pi(100 - y^2)$. You can substitute $y = 0, 1, 2, \ldots, 9$ to compute the volumes of the ten discs.

Now suppose you use n discs to approximate the upper hemisphere. Then the height of each disc equals $\frac{10}{n}$, and the volume of a disc y units from the center of the sphere is $V = \pi(100 - y^2) \cdot \frac{10}{n}$. The following computer program adds the volumes of the n discs, then doubles the result. Note that line 80 calculates the volume of the sphere using the formula $V = \frac{4}{3}\pi r^3$.

```
10  LET Y = 0
15  LET V = 0
20  PRINT "HOW MANY DISCS";
25  INPUT N
30  FOR I = 1 TO N
40  LET Y = (I - 1) * 10/N
50  LET V = V + 3.14159 * (100 - Y↑2) * (10/N)
60  NEXT I
70  PRINT "VOLUME OF DISCS IS ";2 * V
80  PRINT "VOLUME OF SPHERE IS ";4/3 * 3.14159 * 10↑3
90  END
```

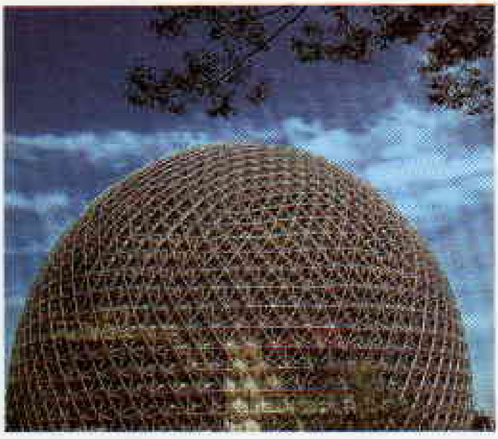
Exercises

1. Use 10 for N and RUN the program. By about what percent does the disc method overapproximate the volume of the sphere?

2. RUN the program to find the total volume of n discs for each value of n.
a. 20 **b.** 50 **c.** 100 **d.** 1000
As n increases, does the approximate volume approach the actual volume?

3. The program uses discs that extend outside the sphere, so it yields approximations greater than the actual volume. To use discs that are inside the sphere, replace line 40 with: `LET Y = I * (10/N)`
   a. RUN the new program for `N = 100`.
   b. Find the average of the result in part (a) and the result in Exercise 2, part (c). Is the average close to the actual volume of the sphere?

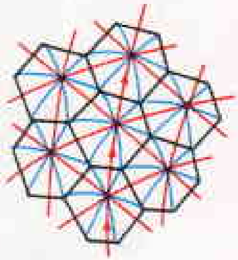
Application — *Geodesic Domes*



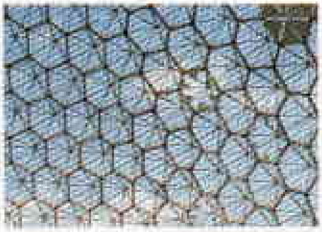
A spherical dome is an efficient way of enclosing space, since a sphere holds a greater volume than any other container with the same surface area. (See Calculator Exercise 2 on page 503.) In 1947, R. Buckminster Fuller patented the *geodesic dome*, a framework made by joining straight pieces of steel or aluminum tubing in a network of triangles. A thin cover of aluminum or plastic is then attached to the tubing.

The segments forming the network are of various lengths, but the vertices are all equidistant from the center of the dome, so that they lie on a sphere. When we follow a chain of segments around the dome, we find that they approximate a circle on this sphere, often a great circle. It is this property that gives the dome design its name: A *geodesic* on any surface is a path of minimum length between two points on the surface, and on a sphere these shortest paths are arcs of great circles.

Though the geodesic dome is very light and has no internal supports, it is very strong, and standardized parts make construction of the dome relatively easy. Domes have been used with success for theaters, exhibition halls, sports arenas, and greenhouses.



The United States Pavilion that Fuller designed for Expo '67 in Montreal uses two domes linked together. The design of this structure is illustrated at the right. The red triangular network is the outer dome, the black hexagons form the inner dome, and the blue segments represent the trusses that tie the two domes together. The arrows mark one of the many chains of segments that form arcs of circles on the dome. You can see all of these features of the structure in the photograph at the right, which shows a view from inside the dome.

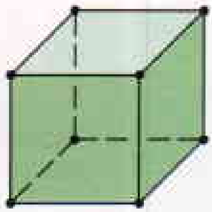


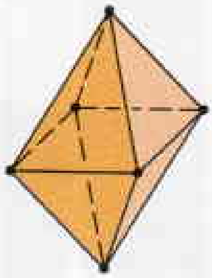
Although a grid of hexagons will interlock nicely to cover the plane, they cannot interlock to cover a sphere unless twelve of the hexagons are changed to pentagons. (The reason for this is given in the exercises on the next page.)

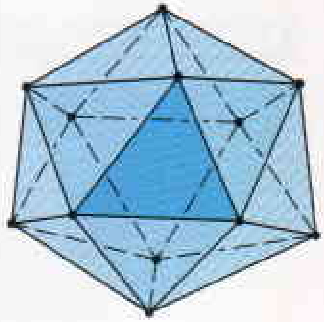
Exercises

For any solid figure with polygons for faces, Euler's formula, $F + V - E = 2$, must hold. In this formula, F, V, and E stand for the number of faces, vertices, and edges, respectively, that the figure has.

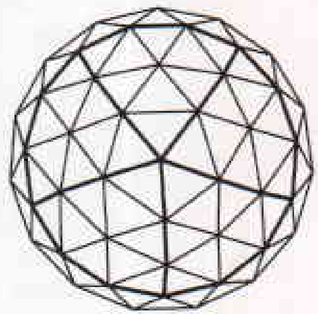
1. Verify Euler's formula for each figure below.
   a. Cube **b.** Octahedron **c.** Icosahedron (20 faces)







2. If each edge of the icosahedron above is trisected and the trisection points are "popped out" to the surface of the circumscribing sphere, one of the many possible geodesic domes is formed. By subdividing the edges of the icosahedron into more than three parts, the resulting geodesic dome is even more spherelike, as shown. In the diagram, find a group of equilateral triangles that cluster to form **(a)** a hexagon, and **(b)** a pentagon.



3. In this exercise, Euler's formula will be used to show that **(a)** the dome's framework *cannot* consist of hexagons only, and **(b)** the framework *can* consist of hexagons plus exactly 12 pentagons.
   a. Use an indirect proof and assume that the framework has n faces, all hexagons. Thus $F = n$. To find V, the number of vertices on the framework, notice that each hexagon contributes 6 vertices, but each vertex is shared by 3 hexagons. Thus $V = \frac{6n}{3}$. To find E, the number of edges of the framework, notice that each hexagon contributes 6 edges, but each edge is shared by 2 hexagons. Thus $E = \frac{6n}{2}$. According to Euler's Formula: $F + V - E$ must equal 2. Does it? What does this contradiction tell you?
   b. Suppose that 12 of the n faces of the framework are pentagons. Show that $V = \frac{6n - 12}{3}$ and that $E = \frac{6n - 12}{2}$. Then use algebra to show that $F + V - E = 2$. Since Euler's formula is satisfied, a dome framework can be constructed when n faces consist of 12 pentagons and $n - 12$ hexagons.

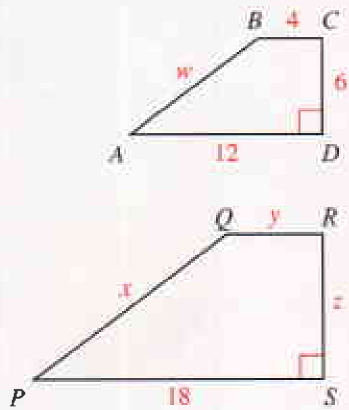
Biographical Note — R. Buckminster Fuller

The early curiosity shown by R. Buckminster Fuller (1895–1983) about the world around him led to a life of invention and philosophy. As a mathematician he made many contributions to the fields of engineering, architecture, and cartography. His ultimate goal was always "to do more with less." Thus his discoveries often had economic and ecological implications.

Fuller's inventions include the geodesic dome (see pages 505 and 506), the 3-wheeled Dymaxion car, and the Dymaxion Air-ocean World Map on which he was able to project the spherical earth as a flat surface without any visible distortions. He also designed other structures that were based upon triangles and circles instead of the usual rectangular surfaces.

Mixed Review Exercises

Trapezoid *ABCD* is similar to trapezoid *PQRS*.



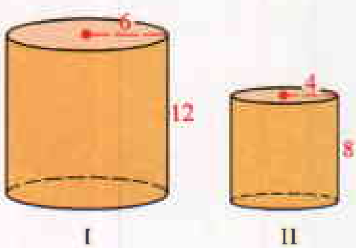
1. Find the scale factor of the trapezoids.
2. Draw an altitude from point B and use the Pythagorean Theorem to find the value of w.
3. Find the values of x, y, and z.
4. **a.** Find the perimeter of each trapezoid.
   b. Find the ratio of the perimeters.
   c. Compare the ratio of the perimeters to the scale factor you found in Exercise 1.
5. **a.** Find the area of each trapezoid.
   b. Find the ratio of the areas.
   c. Compare the ratio of the areas to the scale factor you found in Exercise 1.
6. For each of the following, complete the statement: All __?__ are similar. Classify the statement as true or false.
   a. squares
   b. rectangles
   c. circles
   d. rhombuses
   e. right triangles
   f. equilateral triangles
   g. regular pentagons
   h. isosceles trapezoids

12-5 *Areas and Volumes of Similar Solids*

One of the best-known attractions in The Hague, the Netherlands, is a unique miniature city, Madurodam, consisting of five acres of carefully crafted reproductions done on a scale of 1:25. Everything in this model city works, including the two-mile railway network, the canal locks, the harbor fireboats, and the nearly 50,000 tiny street lights. In this section you will study the relationship between scale factors of *similar solids* and their areas and volumes.

Similar solids are solids that have the same shape but not necessarily the same size. It's easy to see that all spheres are similar. To decide whether two other solids are similar, determine whether bases are similar and corresponding lengths are proportional.

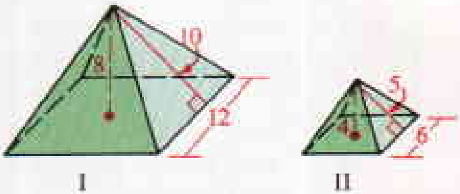
Right cylinders



The bases are similar because all circles are similar. The lengths are proportional because $\frac{6}{4} = \frac{12}{8}$.

So the solids are similar with scale factor $\frac{3}{2}$.

Regular square pyramids



The bases are similar because all squares are similar. The lengths are proportional because $\frac{12}{6} = \frac{8}{4} = \frac{10}{5}$.

So the solids are similar with scale factor $\frac{2}{1}$.

The table below shows the ratios of the perimeters, areas, and volumes for both pairs of similar solids shown on page 508. Notice the relationship between the scale factor and the ratios in each column.

	Cylinders I and II	**Pyramids I and II**
Scale factor	$\frac{3}{2}$	$\frac{2}{1}$
$\frac{\text{Base perimeter (I)}}{\text{Base perimeter (II)}}$	$\frac{2\pi \cdot 6}{2\pi \cdot 4} = \frac{6}{4}$, or $\frac{3}{2}$	$\frac{4 \cdot 12}{4 \cdot 6} = \frac{12}{6}$, or $\frac{2}{1}$
$\frac{\text{L.A. (I)}}{\text{L.A. (II)}}$	$\frac{2\pi \cdot 6 \cdot 12}{2\pi \cdot 4 \cdot 8} = \frac{9}{4}$, or $\frac{3^2}{2^2}$	$\frac{\frac{1}{2} \cdot 48 \cdot 10}{\frac{1}{2} \cdot 24 \cdot 5} = \frac{4}{1}$, or $\frac{2^2}{1^2}$
$\frac{\text{Volume (I)}}{\text{Volume (II)}}$	$\frac{\pi \cdot 6^2 \cdot 12}{\pi \cdot 4^2 \cdot 8} = \frac{27}{8}$, or $\frac{3^3}{2^3}$	$\frac{\frac{1}{3} \cdot 12^2 \cdot 8}{\frac{1}{3} \cdot 6^2 \cdot 4} = \frac{8}{1}$, or $\frac{2^3}{1^3}$

The results shown in the table above are generalized in the following theorem. (See Exercises 22–27 for proofs.)

Theorem 12-11

If the scale factor of two similar solids is $a:b$, then

(1) the ratio of corresponding perimeters is $a:b$.

(2) the ratio of the base areas, of the lateral areas, and of the total areas is $a^2:b^2$.

(3) the ratio of the volumes is $a^3:b^3$.

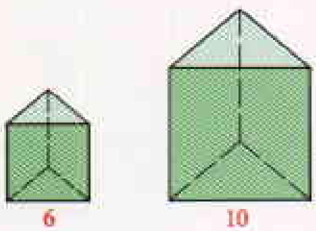
Example For the similar solids shown, find the ratios of the **(a)** base perimeters, **(b)** lateral areas, and **(c)** volumes.

Solution The scale factor is 6:10, or 3:5.

a. Ratio of base perimeters = 3:5

b. Ratio of lateral areas = $3^2:5^2 = 9:25$

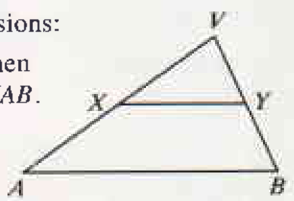
c. Ratio of volumes = $3^3:5^3 = 27:125$



Theorem 12-11 is the three-dimensional counterpart of Theorem 11-7 on page 457. (Take a minute to compare these theorems.) There is a similar relationship between the two cases shown below.

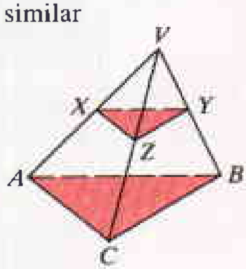
In two dimensions:

If $\overline{XY} \parallel \overline{AB}$, then $\triangle VXY \sim \triangle VAB$.



In three dimensions:

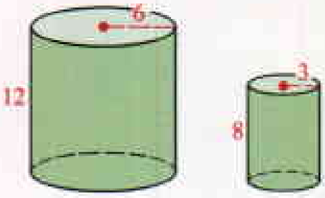
If plane $XYZ \parallel$ plane ABC, then $V\text{-}XYZ \sim V\text{-}ABC$.



Classroom Exercises

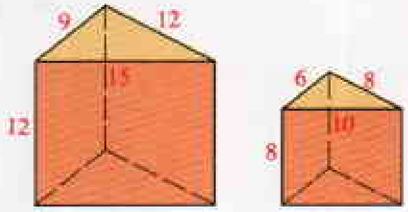
Tell whether the solids in each pair are similar. Explain your answer.

1.



Right cylinders

2.



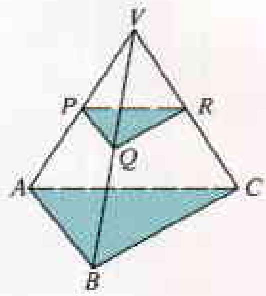
Right prisms

3. For the prisms in Exercise 2, find the ratios of:
   a. the lateral areas b. the total areas c. the volumes
4. Two spheres have diameters 24 cm and 36 cm.
   a. What is the ratio of the areas? b. What is the ratio of the volumes?
5. Two spheres have volumes 2π m^3 and 16π m^3. Find the ratios of:
   a. the volumes b. the diameters c. the areas

Complete the table below, which refers to two similar cones.

	6.	7.	8.	9.	10.	11.
Scale factor	3:4	5:7	?	?	?	?
Ratio of base circumferences	?	?	2:1	?	?	?
Ratio of slant heights	?	?	?	1:6	?	?
Ratio of lateral areas	?	?	?	?	4:9	?
Ratio of total areas	?	?	?	?	?	?
Ratio of volumes	?	?	?	?	?	8:125

12. Plane PQR is parallel to the base of the pyramid and bisects the altitude. Find the following ratios.
    a. The perimeter of $\triangle PQR$ to the perimeter of $\triangle ABC$
    b. The lateral area of the top part of the pyramid to the lateral area of the whole pyramid
    c. The lateral area of the top part of the pyramid to the lateral area of the bottom part
    d. The volume of the top part of the pyramid to the volume of the bottom part



13. Find each ratio in Exercise 12 if the height of the top pyramid is 3 cm and the height of the whole pyramid is 5 cm.

Written Exercises

A

1. Two cones have radii 6 cm and 9 cm. The heights are 10 cm and 15 cm, respectively. Are the cones similar?
2. The heights of two right prisms are 18 ft and 30 ft. The bases are squares with sides 8 ft and 15 ft, respectively. Are the prisms similar?
3. Two similar cylinders have radii 3 and 4. Find the ratios of the following:
   a. heights **b.** base circumferences **c.** lateral areas **d.** volumes
4. Two similar pyramids have heights 12 and 18. Find the ratios of the following:
   a. base areas **b.** lateral areas **c.** total areas **d.** volumes
5. Assume that the Earth and the moon are smooth spheres with diameters 12,800 km and 3,200 km, respectively. Find the ratios of the following:
   a. lengths of their equators **b.** areas **c.** volumes
6. Two similar cylinders have lateral areas 81π and 144π. Find the ratios of:
   a. the heights **b.** the total areas **c.** the volumes
7. Two similar cones have volumes 8π and 27π. Find the ratios of:
   a. the radii **b.** the slant heights **c.** the lateral areas
8. Two similar pyramids have volumes 3 and 375. Find the ratios of:
   a. the heights **b.** the base areas **c.** the total areas
9. The package of a model airplane kit states that the scale is 1:200. Compare the amounts of paint required to cover the model and the actual airplane. (Assume the paint on the model is as thick as that on the actual airplane.)

10. The scale for a certain model freight train is 1:48. If the model hopper car (usually used for carrying coal) will hold 90 in.^{3} of coal, what is the capacity in cubic feet of the actual hopper car? (*Hint*: See Exercise 10, page 477.)
11. Two similar cones have radii of 4 cm and 6 cm. The total area of the smaller cone is 36π cm^2. Find the total area of the larger cone.

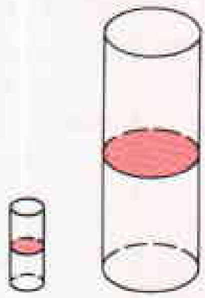
B

12. A diagonal of one cube is 2 cm. A diagonal of another cube is $4\sqrt{3}$ cm. The larger cube has volume 64 cm^3. Find the volume of the smaller cube.
13. Two balls made of the same metal have radii 6 cm and 10 cm. If the smaller ball weighs 4 kg, find the weight of the larger ball to the nearest 0.1 kg.

14. A snow man is made using three balls of snow with diameters 30 cm, 40 cm, and 50 cm. If the head weighs about 6 kg, find the total weight of the snow man. (Ignore the arms, eyes, nose and mouth.)

15. A certain kind of string is sold in a ball 6 cm in diameter and in a ball 12 cm in diameter. The smaller ball costs \$1.00 and the larger one costs \$6.50. Which is the better buy?

16. Construction engineers know that the strength of a column is proportional to the area of its cross section. Suppose that the larger of two similar columns is three times as high as the smaller column.
a. The larger column is __?__ times as strong as the smaller column.
b. The larger column is __?__ times as heavy as the smaller column.
c. Which can support more, *per pound of column material*, the larger or the smaller column?



17. Two similar pyramids have lateral areas 8 ft^2 and 18 ft^2. If the volume of the smaller pyramid is 32 ft^3, what is the volume of the larger pyramid?

18. Two similar cones have volumes 12π and 96π. If the lateral area of the smaller cone is 15π, what is the lateral area of the larger cone?

19. A plane parallel to the base of a cone divides the cone into two pieces. Find the ratios of the following:
a. The areas of the shaded circles
b. The lateral area of the top part of the cone to the lateral area of the whole cone
c. The lateral area of the top part of the cone to the lateral area of the bottom part
d. The volume of the top part of the cone to the volume of the whole cone
e. The volume of the top part of the cone to the volume of the bottom part

20. Redraw the figure for Exercise 19, changing the 9 cm and 3 cm dimensions to 10 cm and 4 cm, respectively. Then find the five ratios described in Exercise 19.

21. A pyramid with height 15 cm is separated into two pieces by a plane parallel to the base and 6 cm above it. What are the volumes of these two pieces if the volume of the original pyramid is 250 cm^3?

The purpose of Exercises 22–27 is to prove Theorem 12-11 for some similar solids.

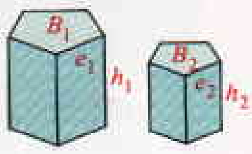
22. Two spheres have radii a and b. Prove that the ratio of the areas is $a^2:b^2$.

23. Two spheres have radii a and b. Prove that the ratio of the volumes is $a^3:b^3$.

24. Two similar cones have radii r_1 and r_2 and heights h_1 and h_2. Prove that the ratio of the volumes is $h_1{}^3:h_2{}^3$.

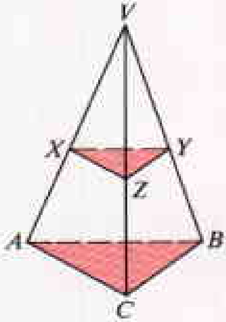
25. Two similar cones have radii r_1 and r_2 and slant heights l_1 and l_2. Prove that the ratio of the lateral areas is $r_1{}^2:r_2{}^2$.

26. The bases of two similar right prisms are regular pentagons with base edges e_1 and e_2 and base areas B_1 and B_2. The heights are h_1 and h_2. Prove that the ratio of the lateral areas is $e_1^2 : e_2^2$.



27. Refer to Exercise 26. Prove that the ratio of the volumes of the prisms is $e_1^3 : e_2^3$.

C 28. The purpose of this exercise is to prove that if plane XYZ is parallel to plane ABC, then $V\text{-}XYZ \sim V\text{-}ABC$. To do this, suppose that $VA = k \cdot VX$ and show that every edge of $V\text{-}ABC$ is k times as long as the corresponding edge of $V\text{-}XYZ$. (*Hint*: Use Theorem 3-1.)



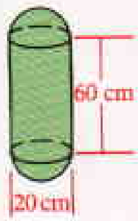
29. A plane parallel to the base of a pyramid separates the pyramid into two pieces with equal volumes. If the height of the pyramid is 12, find the height of the top piece.

Self-Test 2

1. Find the area and volume of a sphere with diameter 6 cm.

2. The volume of a sphere is $\frac{32}{3}\pi$ m^3. Find the area of the sphere.

3. The students of a school decide to bury a time capsule consisting of a cylinder capped by two hemispheres. Find the volume of the time capsule shown.



Ex. 3

4. Find the area of the circle formed when a plane passes 12 cm from the center of a sphere with radius 13 cm.

5. One regular triangular pyramid has base edge 8 and height 6. A similar pyramid has height 4.

a. Find the base edge of the smaller pyramid.

b. Find the ratio of the total areas of the pyramids.

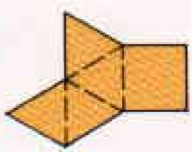
6. The base areas of two similar prisms are 32 and 200, respectively.

a. Find the ratio of their heights. **b.** Find the ratio of their volumes.

Challenge

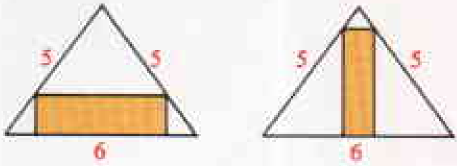
A pattern for a model is shown. Can you tell what it is? To build it, make a large copy of the pattern on stiff paper. Cut along the solid lines, fold along the dashed lines, and tape the edges together.

If you want to make a pattern for a figure, think about the number of faces, their shapes, and how the edges are related. Try to create and build models for a triangular prism, a triangular pyramid, and a cone.

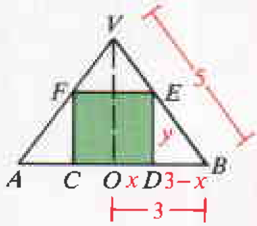


• Calculator Key-In

Each diagram shows a rectangle inscribed in an isosceles triangle with legs 5 and base 6. There are many more such rectangles. Which one has the greatest area?



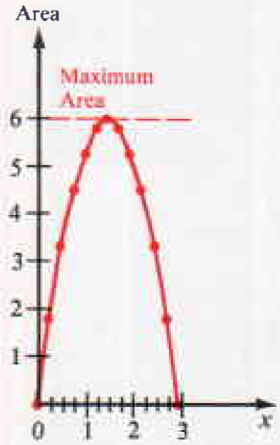
To solve the problem, let $CDEF$ represent any rectangle inscribed in isosceles $\triangle ABV$ with legs 5 and base 6. If we let $OD = x$ and $ED = y$, then the area of the rectangle is $2xy$. Our goal is to express this area in terms of x alone. Then we can find out how the area changes as x changes.



1. In right $\triangle VOB$, $OB = 3$ and $VB = 5$. Thus $VO = 4$ by the Pythagorean Theorem.
2. $\triangle EDB \sim \triangle VOB$ (Why?)
3. $\frac{ED}{VO} = \frac{DB}{OB}$ (Why?)
4. $\frac{y}{4} = \frac{3 - x}{3}$ (By substitution in Step 3)
5. $y = \frac{4}{3}(3 - x)$ (Multiplication Property of =)
6. Area of rectangle: $A = 2xy = 2x \cdot \frac{4}{3}(3 - x) = \frac{8x(3 - x)}{3}$

Use the formula in Step 6 and a calculator to find the area for many values of x. Calculate $3 - x$ first, then multiply by x, then multiply by 8, and divide by 3.

x	Area
0	0
0.25	1.83333
0.5	3.33333
0.75	4.5
1	5.33333
1.25	5.83333
1.5	6
1.75	5.83333
2	5.33333
2.25	4.5
2.50	3.33333
2.75	1.83333
3	0



The table was used to make a graph showing how the area varies with x. Both the table and the graph suggest that the greatest area, 6 square units, occurs when $x = 1.5$.

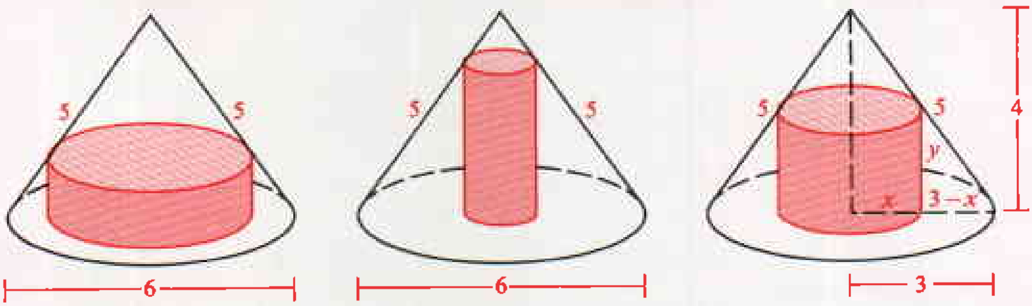
Exercises

Suppose the original triangle had sides 5, 5, and 8 instead of 5, 5, and 6.

1. Draw a diagram. Then show that $A = \dfrac{3x(4 - x)}{2}$.
2. Find the value of x for which the greatest area occurs.

♦ Computer Key-In

A rectangle is inscribed in an isosceles triangle with legs 5 and base 6 and the triangle is rotated in space about the altitude to the base. The resulting figure is a cylinder inscribed in a cone with diameter 6 and slant height 5, as shown below. Which of the cylinders such as these has the greatest volume?



The diagram at the right above shows a typical inscribed cylinder. Using similar triangles, we have the proportion $\frac{y}{4} = \frac{3 - x}{3}$. Thus $y = \frac{4}{3}(3 - x)$.

The volume of the cylinder is found as follows:

$$V = \pi x^2 y = \pi x^2 \cdot \tfrac{4}{3}(3 - x) \approx \tfrac{4}{3}(3.14159)x^2(3 - x)$$

The following program in BASIC evaluates V for various values of x.

```
10  PRINT "X", "VOLUME"
20  FOR X = 0 TO 3 STEP 0.25
30  LET V = 4/3 * 3.14159 * X↑2 * (3 - X)
40  PRINT X, V
50  NEXT X
60  END
```

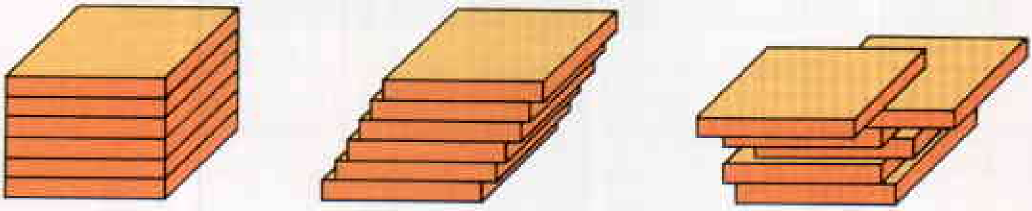
Exercises

1. RUN the program. Make a graph that shows how the volume varies with x. For what value of x did you find the greatest volume?
2. Suppose the original triangle has sides 5, 5, and 8 instead of 5, 5, and 6. Rotate the triangle in space about the altitude to the base.
   a. Draw a diagram. Show that $V = \frac{3}{4}\pi x^2(4 - x)$.
   b. Change lines 20 and 30 of the program and RUN the revised program to find the value of x for which the greatest volume occurs.

Extra

Cavalieri's Principle

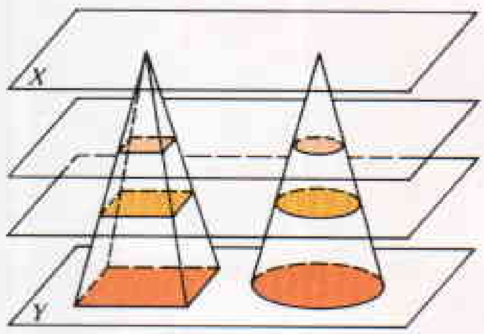
Suppose you have a right rectangular prism and divide it horizontally into thin rectangular slices. The base of each rectangular slice, or *cross section*, has the same area as the base of the prism. If you rearrange the slices, the total volume of the slices does not change.



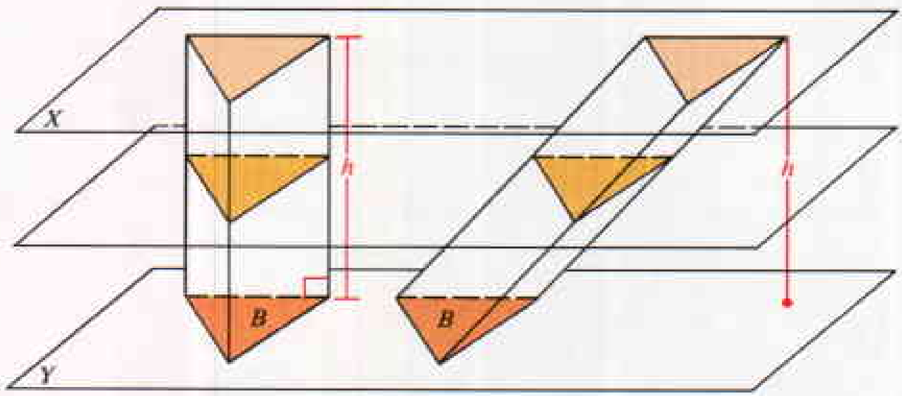
Bonaventura Cavalieri (1598–1647), an Italian mathematician, used this idea to compare the volumes of solids. His conclusion is known as *Cavalieri's Principle*.

Cavalieri's Principle

If two solids lying between parallel planes have equal heights and all cross sections at equal distances from their bases have equal areas, then the solids have equal volumes.



Using Cavalieri's Principle you can find the volume of an oblique prism. Consider a right triangular prism and an oblique prism that have the same base and height.



By Theorem 12-2, the volume of the right prism is $V = Bh$. Every cross section of each prism has the same area as that prism's base. Since the base areas are equal, the corresponding cross sections of the two prisms have equal areas. Therefore by Cavalieri's Principle, the volume of the oblique prism also is $V = Bh$.

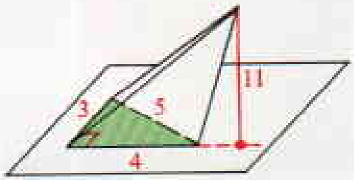
You can use similar reasoning to show that the volume formulas given for a regular pyramid, right cylinder, and right cone hold true for the corresponding oblique solids.

$V = Bh$ for *any* prism or cylinder
$V = \frac{1}{3}Bh$ for *any* pyramid or cone

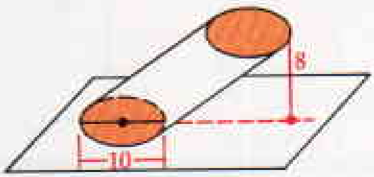
Exercises

Find the volume of the solid shown with the given altitude.

1.

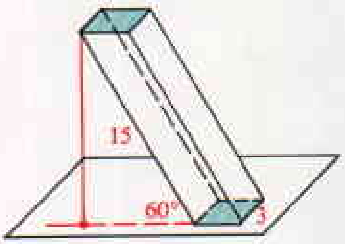


2.

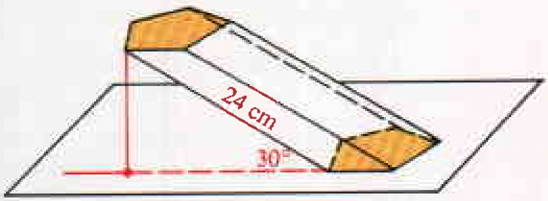


3. Find the volume of an oblique cone with radius 4 and height 3.5.

4. The oblique square prism shown below has base edge 3. A lateral edge that is 15 makes a 60° angle with the plane containing the base. Find the exact volume.



Ex. 4



Ex. 5

5. The volume of the oblique pentagonal prism shown above is 96 cm^3. A lateral edge that is 24 cm makes a 30° angle with the plane containing the base. Find the area of the base.

6. Refer to the justification of the formula for the volume of a sphere given on pages 498–499. How does Cavalieri's Principle justify the statement that the volume of the sphere is equal to the difference between the volumes of the cylinder and the double cone?

Chapter Summary

1. The list below summarizes area and volume formulas for solids. The cylinder formulas are special cases of the prism formulas with $p = 2\pi r$ and $B = \pi r^2$. Also the cone formulas are special cases of the pyramid formulas with the same substitutions for p and B. To find the total area of each of the four solids, add lateral area to the area of the base(s).

Right prism	L.A. $= ph$	$V = Bh$
Right cylinder	L.A. $= 2\pi rh$	$V = \pi r^2 h$
Regular pyramid	L.A. $= \frac{1}{2}pl$	$V = \frac{1}{3}Bh$
Right cone	L.A. $= \pi rl$	$V = \frac{1}{3}\pi r^2 h$
Sphere	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

2. If the scale factor of two similar solids is $a:b$, then
   a. the ratio of corresponding perimeters is $a:b$.
   b. the ratio of corresponding areas is $a^2:b^2$.
   c. the ratio of the volumes is $a^3:b^3$.

Chapter Review

1. In a right prism, each _?_ is also an altitude. **12–1**
2. Find the lateral area of a right octagonal prism with height 12 and base edge 7.
3. Find the total area and volume of a rectangular solid with dimensions 8, 6, and 5.
4. A right square prism has base edge 9 and volume 891. Find the total area.
5. Find the volume of a regular triangular pyramid with base edge 8 and height 10. **12–2**
6. A regular pentagonal pyramid has base edge 6 and lateral edge 5. Find the slant height and the lateral area.

A regular square pyramid has base edge 30 and total area 1920.

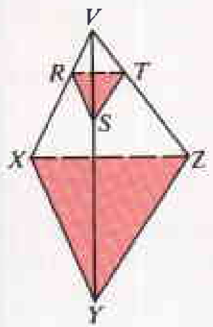
7. Find the area of the base, the lateral area, and the slant height.
8. Find the height and the volume of the pyramid.
9. Find the lateral area and the total area of a cylinder with radius 4 and height 3. **12–3**
10. Find the lateral area, total area, and volume of a cone with radius 6 cm and slant height 10 cm.
11. A cone has volume 8π cm^3 and height 6 cm. Find its slant height.
12. The radius of a cylinder is doubled and its height is halved. How does the volume change?

13. A sphere has radius 7 m. Use $\pi \approx \frac{22}{7}$ to find the approximate area of the sphere. 12–4

14. Find, in terms of π, the volume of a sphere with diameter 12 ft.

15. Find the volume of a sphere with area 484π cm^2.

Plane *RST* ∥ plane *XYZ* and *VS*:*VY* = 1:3.



12–5

16. $\dfrac{\text{perimeter of } \triangle RST}{\text{perimeter of } \triangle XYZ} = \underline{\quad ? \quad}$

17. $\dfrac{\text{total area of small pyramid}}{\text{total area of large pyramid}} = \underline{\quad ? \quad}$

18. $\dfrac{\text{volume of small pyramid}}{\text{volume of bottom part}} = \underline{\quad ? \quad}$

19. Two similar cylinders have lateral areas 48π and 27π. Find the ratio of their volumes.

Chapter Test

1. Find the volume and the total area of a cube with edge $2k$.
2. A regular square pyramid has base edge 3 cm and volume 135 cm^2. Find the height.
3. A cone has radius 8 and height 6. Find the volume.
4. Find the lateral area and the total area of the cone in Exercise 3.
5. A right triangular prism has height 20 and base edges 5, 12, and 13. Find the total area.
6. Find the volume of the prism in Exercise 5.
7. A cylinder has radius 6 cm and height 4 cm. Find the lateral area.
8. Find the volume of the cylinder in Exercise 7.
9. A regular square pyramid has lateral area 60 m^2 and base edge 6 m. Find the volume.
10. A sphere has radius 6 cm. Find the area and the volume.
11. Two cones have radii 12 cm and 18 cm, and have slant heights 18 cm and 24 cm. Are the cones similar? Explain.
12. A regular pyramid has height 18 and total area 648. A similar pyramid has height 6. Find the total area of the smaller pyramid.
13. The volumes of two similar rectangular solids are 1000 cm^3 and 64 cm^3. What is the ratio of their lateral areas?
14. A cone and a cylinder each have radius 3 and height 4. Find the ratio of their volumes and of their lateral areas.
15. Find the volume of a sphere with area 9π.
16. A cylinder with radius 7 has total area 168π cm^2. Find its height.

Preparing for College Entrance Exams

Strategy for Success

Questions on college entrance exams often require knowledge of areas and volumes. Be sure that you know all the important formulas developed in Chapters 11 and 12. To avoid doing unnecessary calculations, be sure to read the directions to find out whether answers may be expressed in terms of π.

Indicate the best answer by writing the appropriate letter.

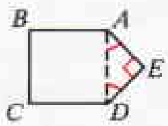
1. A cone has volume 320π and height 15. Find the total area.
(A) 200π (B) 368π (C) 264π (D) 136π (E) 320π

2. Two equilateral triangles have perimeters 6 and $9\sqrt{3}$. The ratio of their areas is:
(A) $2:3\sqrt{3}$ (B) $2\sqrt{3}:9$ (C) $4:27$ (D) $4:9$ (E) $8:81\sqrt{3}$

3. A sphere has volume 288π. Its diameter is:
(A) $12\sqrt{6}$ (B) $6\sqrt{2}$ (C) 6 (D) $12\sqrt{2}$ (E) 12

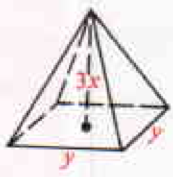
4. *RSTW* is a rhombus with $m\angle R = 60$ and $RS = 4$. If X is the midpoint of $\overline{RS}$, find the area of trapezoid *SXWT*.
(A) 12 (B) 16 (C) $8\sqrt{3}$ (D) $6\sqrt{3}$ (E) $16 - 2\sqrt{2}$

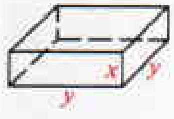
5. If *ABCD* is a square and $AE = y$, the area of *ABCDE* is
(A) $\frac{5}{4}y^2$ (B) $\frac{5}{2}y^2$ (C) $3y^2$
(D) $(4 + \frac{1}{2}\sqrt{3})y^2$ (E) $(\frac{1}{2} + \sqrt{2})y^2$



Compare the quantity in Column A with that in Column B. Select:
(A) if the quantity in Column A is greater;
(B) if the quantity in Column B is greater;
(C) if the two quantities are equal;
(D) if the relationship cannot be determined from the information given.

Column A	Column B
6. volume of square pyramid	volume of square prism





7. area of triangle	area of sector

Cumulative Review: Chapters 1–12

For Exercises 1–9 classify each statement as true or false.

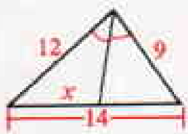
A

1. No more than one plane contains two given intersecting lines.
2. The conditional "p only if q" is equivalent to "if p, then q."
3. If the vertex angle of an isosceles triangle has measure j, then the measure of a base angle is $180 - 2j$.
4. In $\triangle RST$, if $m\angle R = 48$ and $m\angle S = 68$, then $RT > RS$.
5. If right $\triangle JEH$ has hypotenuse $\overline{JE}$, then $\tan J = \frac{JH}{EH}$.
6. It is possible to construct an angle of measure 105.
7. The area of a triangle with sides 3, 3, and 2 is $4\sqrt{2}$.
8. When a square is circumscribed about a circle, the ratio of the areas is $4:\pi$.
9. A triangle with sides of length $\sqrt{3}$, 2, and $\sqrt{7}$ is a right triangle.

B

10. In $\square JKLM$, $m\angle J = \frac{3}{2}x$ and $m\angle L = x + 17$. Find the numerical measure of $\angle K$.
11. Given: $\overline{WZ} \perp \overline{ZY}$; $\overline{WX} \perp \overline{XY}$; $\overline{WX} \cong \overline{YZ}$
    Prove: $\overline{WZ} \parallel \overline{XY}$

12. Prove: If the diagonals of a parallelogram are perpendicular, then the parallelogram must be a rhombus.
13. For $\triangle JKL$ and $\triangle XYZ$ use the following statement:
    "If $\angle J \cong \angle X$ and $\angle K \cong \angle Y$, then $\triangle JKL \sim \triangle XYZ$."
    a. Name the postulate or theorem that justifies the statement.
    b. Write the converse of the statement. Is the converse true or false?
14. Find the value of x in the diagram at the right.



15. $\overline{AB}$ and $\overline{CD}$ are chords of $\odot P$ intersecting at X. If $AX = 7.5$, $BX = 3.2$, $CD = 11$, and $CX > DX$, find CX.
16. Describe each possibility for the locus of points in space that are equidistant from the sides of a $\triangle ABC$ and 4 cm from A.
17. $\overset{\frown}{AB}$ lies on $\odot O$ with $m\overset{\frown}{AB} = 60$. $\odot O$ has radius 8. Find AB.
18. A regular square pyramid has base edge 10 and height 12. Find its total area and volume.
19. A cylinder has a radius equal to its height. The total area of the cylinder is 100π cm^2. Find its volume.
20. A sphere has a diameter of 1.8 cm. Find its surface area to the nearest square centimeter. (Use $\pi \approx 3.14$.)