13 COORDINATE GEOMETRY

Every eighteen seconds, a plane lands at or departs O'Hare International Airport in Chicago. Modern instruments and a knowledge of coordinate geometry enable air traffic controllers to track and direct these planes safely and efficiently.

Geometry and Algebra

Objectives

1. State and apply the distance formula.
2. State and apply the general equation of a circle.
3. State and apply the slope formula.
4. Determine whether two lines are parallel, perpendicular, or neither.
5. Understand the basic properties of vectors.
6. State and apply the midpoint formula.

13-1 *The Distance Formula*

Some of the terms you have used in your study of graphs are reviewed below.

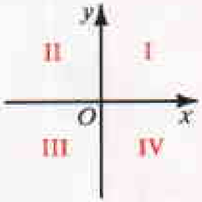
Origin: Point O

Axes: x-axis and y-axis

Quadrants: Regions I, II, III, and IV

Coordinate plane: The plane of the x-axis and the y-axis

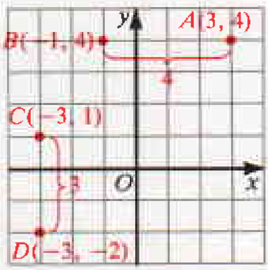
The arrowhead on each axis shows the positive direction.



You can easily find the distance between two points that lie on a horizontal line or on a vertical line.

The distance between A and B is 4.
Using the x-coordinates of A and B:
$|3 - (-1)| = 4$, or $|(-1) - 3| = 4$

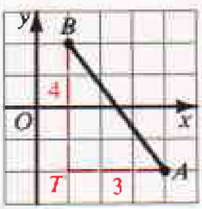
The distance between C and D is 3.
Using the y-coordinates of C and D:
$|1 - (-2)| = 3$, or $|(-2) - 1| = 3$



When two points do not lie on a horizontal or vertical line, you can find the distance between the points by using the Pythagorean Theorem.

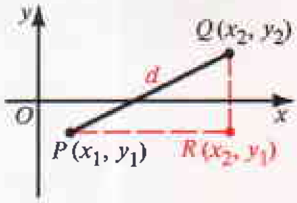
Example 1 Find the distance between points $A(4, -2)$ and $B(1, 2)$.

Solution Draw the horizontal and vertical segments shown. The coordinates of T are $(1, -2)$. Then $AT = 3$, $BT = 4$, $(AB)^2 = 3^2 + 4^2 = 25$, and $AB = 5$.



Using a method suggested by Example 1, you can find a formula for the distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$. First draw a right triangle as shown. The coordinates of R are (x_2, y_1).

$$PR = |x_2 - x_1|;\ QR = |y_2 - y_1|$$
$$d^2 = (PR)^2 + (QR)^2$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Since d represents distance, d cannot be negative.

Theorem 13-1 *The Distance Formula*

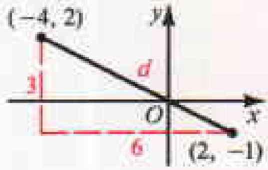
The distance d between points (x_1, y_1) and (x_2, y_2) is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2 Find the distance between points $(-4, 2)$ and $(2, -1)$.

Solution 1 Draw a right triangle. The legs have lengths 6 and 3.

$$d^2 = 6^2 + 3^2 = 36 + 9 = 45$$
$$d = \sqrt{45} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$$

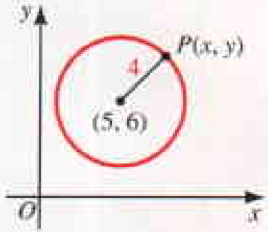


Solution 2 Let (x_1, y_1) be $(-4, 2)$ and (x_2, y_2) be $(2, -1)$.

Then $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$$= \sqrt{(2 - (-4))^2 + ((-1) - 2)^2}$$
$$= \sqrt{6^2 + (-3)^2} = \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$$

You can use the distance formula to find an equation of a circle. Example 3 shows how to do this.

Example 3 Find an equation of a circle with center $(5, 6)$ and radius 4.



Solution Let $P(x, y)$ represent any point on the circle. Since the distance from P to the center is 4,

$$\sqrt{(x - 5)^2 + (y - 6)^2} = 4,$$
$$\text{or } (x - 5)^2 + (y - 6)^2 = 16.$$

Either of these two equations is an equation of the circle, but the second equation is the one usually used.

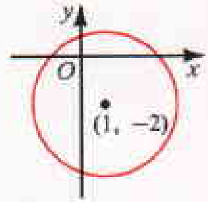
Example 3 can be generalized to give the theorem at the top of the next page.

Theorem 13-2

An equation of the circle with center (a, b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2.$$

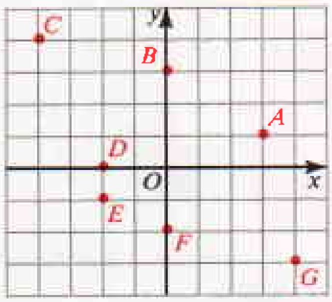
Example 4 Find the center and the radius of the circle with equation $(x - 1)^2 + (y + 2)^2 = 9$. Sketch the graph.



Solution $(x - 1)^2 + (y - (-2))^2 = 3^2$

The center is point $(1, -2)$ and the radius is 3.
The graph is shown at the right.

Classroom Exercises



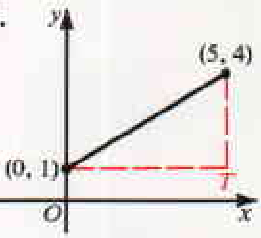
1. What is the x-coordinate of every point that lies on a vertical line through C?
2. Which of the following points lie on a horizontal line through C?

(2, 4)	(2, −4)	(0, 4)
(4, 3)	(15, 4)	(−4, 3)

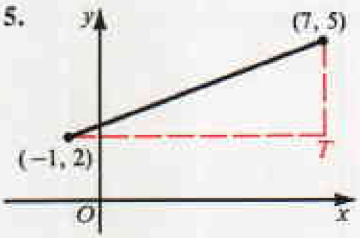
3. Find OD and BF.

In Exercises 4–9 state: a. the coordinates of T
b. the lengths of the legs of the right triangle
c. the length of the segment shown

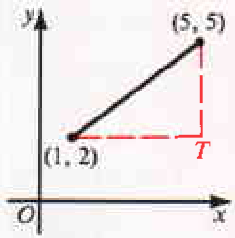
4.



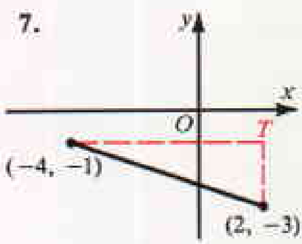
5.



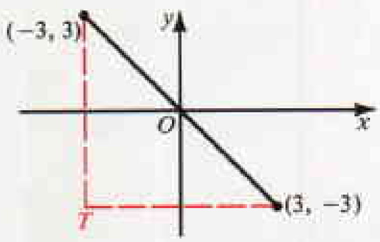
6.



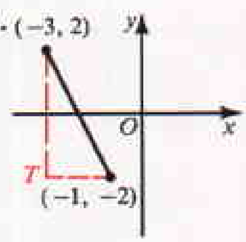
7.



8.



9.



10. Find the distance between the points named. Give all answers in simplest form.

a. (0, 0) and (5, −3) **b.** (3, −2) and (−5, −2) **c.** (4, 4) and (−3, −3)

11. Find the center and the radius of each circle.

a. $(x - 2)^2 + y^2 = 1$ **b.** $(x + 2)^2 + (y - 8)^2 = 16$
c. $x^2 + (y + 5)^2 = 112$ **d.** $(x + 3)^2 + (y + 7)^2 = 14$

12. Find an equation of the circle that has the given center and radius.

a. Center (2, 5); radius 3 **b.** Center (−2, 0); radius 5
c. Center (−2, 3); radius 10 **d.** Center (j, k); radius n

Written Exercises

Find the distance between the two points. If necessary, you may draw graphs but you shouldn't need to use the distance formula.

A **1.** (−2, −3) and (−2, 4) **2.** (3, 3) and (−2, 3)
3. (3, −4) and (−1, −4) **4.** (0, 0) and (3, 4)

Use the distance formula to find the distance between the two points.

5. (−6, −2) and (−7, −5) **6.** (3, 2) and (5, −2)
7. (−8, 6) and (0, 0) **8.** (12, −1) and (0, −6)

Find the distance between the points named. Use any method you choose.

9. (5, 4) and (1, −2) **10.** (−2, −2) and (5, 7)
11. (−2, 3) and (3, −2) **12.** (−4, −1) and (−4, 3)

Given points *A*, *B*, and *C*. Find *AB*, *BC*, and *AC*. Are *A*, *B*, and *C* collinear? If so, which point lies between the other two?

13. $A(0, 3)$, $B(-2, 1)$, $C(3, 6)$ **14.** $A(5, -5)$, $B(0, 5)$, $C(2, 1)$
15. $A(-5, 6)$, $B(0, 2)$, $C(3, 0)$ **16.** $A(3, 4)$, $B(-3, 0)$, $C(-1, 1)$

Find the center and the radius of each circle.

17. $(x + 3)^2 + y^2 = 49$ **18.** $(x + 7)^2 + (y - 8)^2 = \frac{36}{25}$
19. $(x - j)^2 + (y + 14)^2 = 17$ **20.** $(x + a)^2 + (y - b)^2 = c^2$

Write an equation of the circle that has the given center and radius.

21. $C(3, 0)$; $r = 8$ **22.** $C(0, 0)$; $r = 6$
23. $C(-4, -7)$; $r = 5$ **24.** $C(-2, 5)$; $r = \frac{1}{3}$

25. Sketch the graph of $(x - 3)^2 + (y + 4)^2 = 36$.

26. Sketch the graph of $(x - 2)^2 + (y - 5)^2 \leq 9$.

In Exercises 27–32 find and then compare lengths of segments.

B **27.** Show that the triangle with vertices $A(-3, 4)$, $M(3, 1)$, and $Y(0, -2)$ is isosceles.

28. Quadrilateral $TAUL$ has vertices $T(4, 6)$, $A(6, -4)$, $U(-4, -2)$, and $L(-2, 4)$. Show that the diagonals are congruent.

29. Triangles JAN and RFK have vertices $J(-2, -2)$, $A(4, -2)$, $N(2, 2)$, $R(8, 1)$, $F(8, 4)$, and $K(6, 3)$. Show that $\triangle JAN$ is similar to $\triangle RFK$.

30. The vertices of $\triangle KAT$ and $\triangle IES$ are $K(3, -1)$, $A(2, 6)$, $T(5, 1)$, $I(-4, 1)$, $E(-3, -6)$, and $S(-6, -1)$. What word best describes the relationship between $\triangle KAT$ and $\triangle IES$?

31. Find the area of the rectangle with vertices $B(8, 0)$, $T(2, -9)$, $R(-1, -7)$, and $C(5, 2)$.

32. Show that the triangle with vertices $D(0, 0)$, $E(3, 1)$, and $F(-2, 6)$ is a right triangle, then find the area of the triangle.

33. There are twelve points, each with integer coordinates, that are 10 units from the origin. List the points. (*Hint*: Recall the 6, 8, 10 right triangle.)

34. **a.** List twelve points, each with integer coordinates, that are 5 units from $(-8, 1)$.

b. Find an equation of the circle containing these points.

In Exercises 35–38 find an equation of the circle described and sketch the graph.

35. The circle has center $(0, 6)$ and passes through point $(6, 14)$.

36. The circle has center $(-2, -4)$ and passes through point $(3, 8)$.

37. The circle has diameter $\overline{RS}$ where R is $(-3, 2)$ and S is $(3, 2)$.

38. The circle has center (p, q) and is tangent to the x-axis.

39. **a.** Find the radii of the circles

$$x^2 + y^2 = 25 \text{ and } (x - 9)^2 + (y - 12)^2 = 100.$$

b. Find the distance between the centers of the circles.

c. Explain why the circles must be externally tangent.

d. Sketch the graphs of the circles.

40. **a.** Find the radii of the circles

$$x^2 + y^2 = 2 \text{ and } (x - 3)^2 + (y - 3)^2 = 32.$$

b. Find the distance between the centers of the circles.

c. Explain why the circles must be internally tangent.

d. Sketch the graphs of the circles.

41. Discover and prove something about the quadrilateral with vertices $R(-1, -6)$, $A(1, -3)$, $Y(11, 1)$, and $J(9, -2)$.

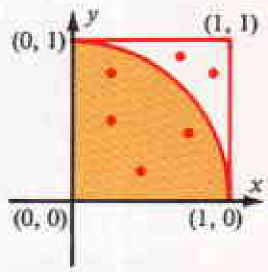
42. Discover and prove two things about the triangle with vertices $K(-3, 4)$, $M(3, 1)$, and $J(-6, -2)$.

C **43.** It is known that $\triangle GHM$ is isosceles. G is point $(-2, -3)$, H is point $(-2, 7)$, and the x-coordinate of M is 4. Find all five possible values for the y-coordinate of M.

44. Find the coordinates of the point that is equidistant from $(-2, 5)$, $(8, 5)$, and $(6, 7)$.

45. Find the center and the radius of the circle $x^2 + 4x + y^2 - 8y = 16$. (*Hint*: Express the given equation in the form
$$(x - a)^2 + (y - b)^2 = r^2.)$$

♦ Computer Key-In

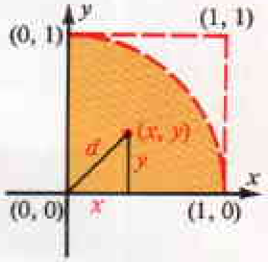


The graph shows a quarter-circle inscribed in a square with area 1. If points are picked at random inside the square, some of them will also be inside the quarter-circle. Let n be the number of points picked inside the square and let q be the number of these points that fall inside the quarter-circle. If many, many points are picked at random inside the square, the following ratios are approximately equal:

$$\frac{\text{Area of quarter-circle}}{\text{Area of square}} \approx \frac{q}{n}$$

$$\frac{\text{Area of quarter-circle}}{1} \approx \frac{q}{n}$$

$$\text{Area of whole circle} \approx 4 \times \frac{q}{n}$$



Any point (x, y) in the square region has coordinates such that $0 < x < 1$ and $0 < y < 1$. (Note that this restriction excludes points on the boundaries of the square.) A computer can pick a random point inside the unit square by choosing two random numbers x and y between 0 and 1. We let d be the distance from O to the point (x, y). By the Pythagorean Theorem, $d = \sqrt{x^2 + y^2}$. Do you see that if $d < 1$, the point lies inside the quarter-circle?

Exercises

1. Write a computer program to do all of the following:
   a. Choose n random points (x, y) inside the unit square.
   b. Using the distance formula test each point chosen to see whether it lies inside the quarter-circle.
   c. Count the number of points (q) which *do* lie inside the quarter-circle.
   d. Print out the value of $4 \times \frac{q}{n}$.

2. RUN your program for $n = 100$, $n = 500$, and $n = 1000$.

3. Calculate the area of the circle, using the formula given on page 446. Compare this result with your computer approximations.

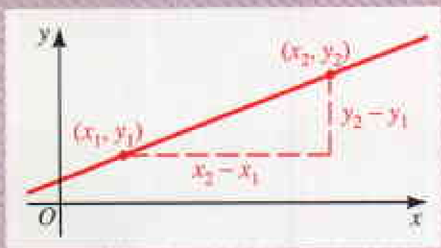
13-2 *Slope of a Line*

The effect of steepness, or *slope*, must be considered in a variety of everyday situations. Some examples are the grade of a road, the pitch of a roof, the incline of a wheelchair ramp, and the tilt of an unloading platform, such as the one at a paper mill in Maine shown in the photograph at the right. In this section, the informal idea of steepness is generalized and made precise by the mathematical concept of *slope of a line through two points*.

Informally, slope is the ratio of the *change in y* (vertical change) to the *change in x* (horizontal change). The **slope,** denoted by m, of the nonvertical line through the points (x_1, y_1) and (x_2, y_2) is defined as follows:

$$\text{slope } m = \frac{y_2 - y_1}{x_2 - x_1}$$

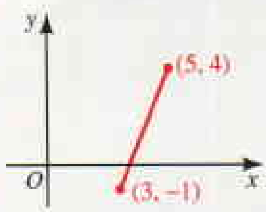
$$= \frac{\text{change in } y}{\text{change in } x}$$



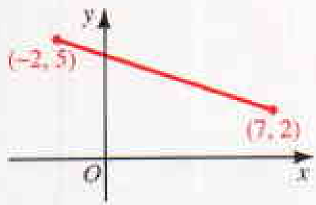
When you are given several points on a line you can use any two of them to compute the slope. Furthermore, the slope of a line does not depend on the order in which the points are chosen because $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

Example 1 Find the slope of each segment.

a.



b.



Solution **a.** $\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{5 - 3} = \frac{5}{2}$ **b.** $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{7 - (-2)} = \frac{-3}{9} = -\frac{1}{3}$

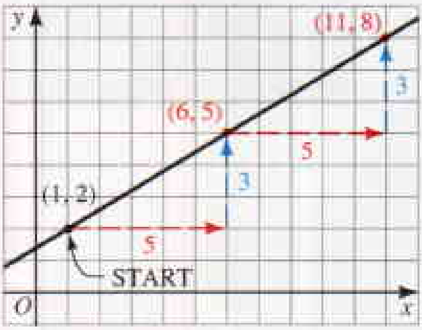
Example 2 Sketch each line described, showing several points on the line.

a. The line passes through (1, 2) and has slope $\frac{3}{5}$.

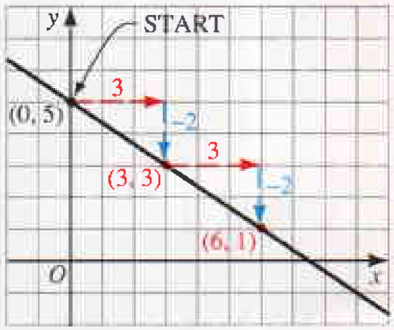
b. The line passes through (0, 5) and has slope $-\frac{2}{3}$.

Solution

a. Since $\frac{\text{change in } y}{\text{change in } x} = \frac{3}{5}$, every horizontal change of 5 units is matched by a vertical change of 3 units. Start at (1, 2), move 5 units to the right and 3 units up.



b. Since $\frac{\text{change in } y}{\text{change in } x} = -\frac{2}{3} = \frac{-2}{3}$, every horizontal change of 3 units is matched by a vertical change of 2 units. Start at (0, 5), move 3 units to the right and 2 units down.

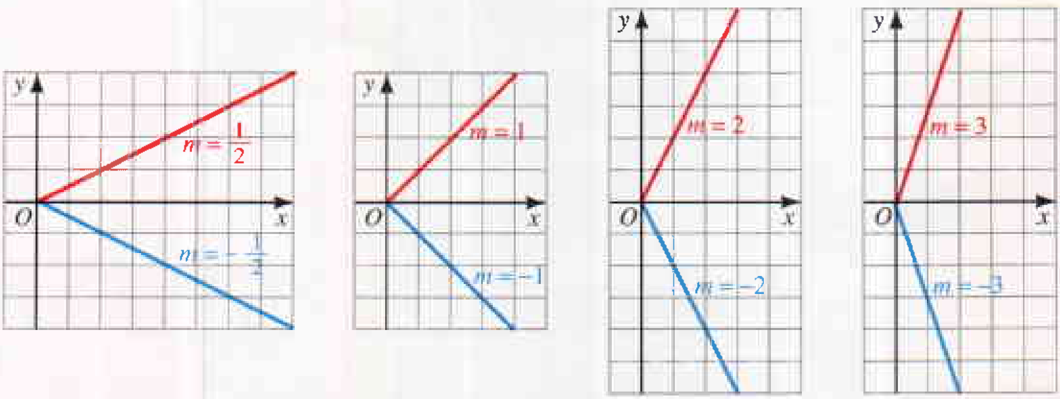


The examples above and the diagrams below illustrate the following facts.

Lines with positive slope rise to the right.

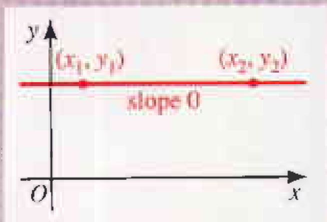
Lines with negative slope fall to the right.

The greater the absolute value of a line's slope, the steeper the line.



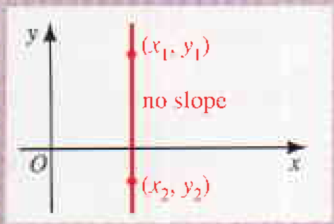
The diagrams below explain why the following are true.

The slope of a horizontal line is zero.



Since $y_1 = y_2$,
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0.$

The slope of a vertical line is not defined.

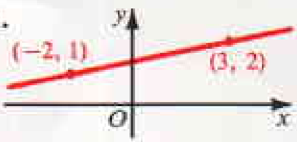


Since $x_1 = x_2$,
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0}$, which is not defined.

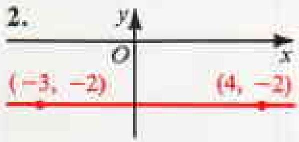
Classroom Exercises

Find the slope of the line.

1.



2.

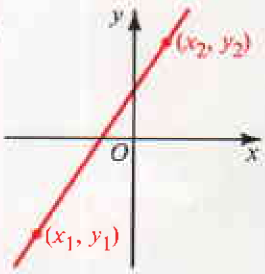


3.

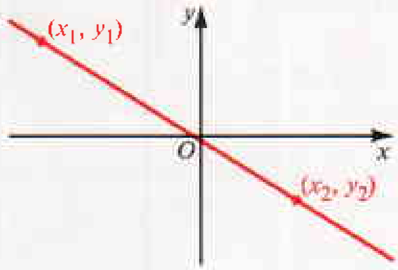
Tell whether each expression is positive or negative for the line shown:

a. $y_2 - y_1$ **b.** $x_2 - x_1$ **c.** $\frac{y_2 - y_1}{x_2 - x_1}$

4.



5.



6. Does the slope of the line appear to be positive, negative, zero, or not defined?

a.

b.

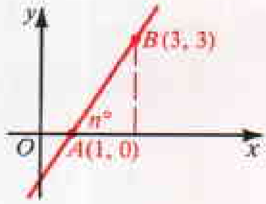
c.

d.

7. a. Find the slope of $\overleftrightarrow{AB}$.

b. Find tan $n°$.

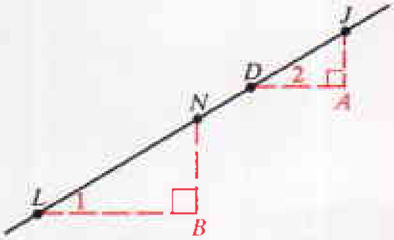
c. Consider the statement: If a line with positive slope makes an acute angle of $n°$ with the x-axis, then the slope of the line is tan $n°$. Do you think this statement is true or false? Explain.



8. This exercise provides a geometric method of justifying the fact that you can use any two points on a line to determine the slope of the line. Horizontal and vertical segments have been drawn as shown. Supply the reason for each step.

Key steps of proof:

1. $\angle B \cong \angle A$
2. $\angle 1 \cong \angle 2$
3. $\triangle LBN \sim \triangle DAJ$
4. $\frac{BN}{AJ} = \frac{LB}{DA}$, or $\frac{BN}{LB} = \frac{AJ}{DA}$
5. The slope of $\overline{LN}$ equals $\frac{BN}{LB}$, and the slope of $\overline{DJ}$ equals $\frac{AJ}{DA}$.
6. Slope of $\overline{LN}$ = slope of $\overline{DJ}$



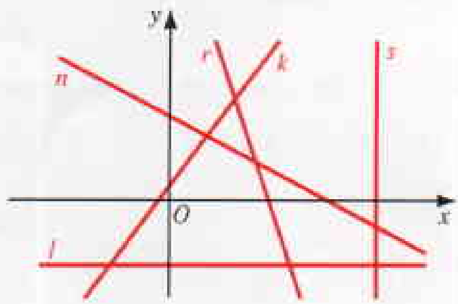
Written Exercises

A

1. Name each line in the figure whose slope is:

a. positive
b. negative
c. zero
d. not defined

2. What can you say about the slope of **(a)** the x-axis? and **(b)** the y-axis?



Find the slope of the line through the points named. If the slope is not defined, write *not defined*.

3. (1, 2); (3, 4)
4. (1, 2); (−2, −5)
5. (1, 2); (−2, 5)
6. (0, 0); (5, 1)
7. (7, 2); (2, 7)
8. (3, 3); (3, 7)
9. (6, −6); (−6, −6)
10. (6, −6); (4, 3)
11. (−4, −3); (−6, −6)

Find the slope and length of $\overline{AB}$.

12. $A(3, -1)$, $B(5, -7)$
13. $A(-3, -2)$, $B(7, -6)$
14. $A(8, -7)$, $B(-3, -5)$
15. $A(0, -9)$, $B(8, -3)$

In Exercises 16–19 a point P on a line and the slope of the line are given. Sketch the line and find the coordinates of two other points on the line.

16. $P(-2, 1)$; slope $= \frac{1}{3}$

17. $P(-3, 0)$; slope $= \frac{2}{5}$

18. $P(2, 4)$; slope $= -\frac{3}{2}$

19. $P(0, -5)$; slope $= -\frac{1}{4}$

In Exercises 20 and 21 show that points P, Q, and R are collinear by showing that $\overline{PQ}$ and $\overline{QR}$ have the same slope.

20. $P(-1, 3)$ $Q(2, 7)$ $R(8, 15)$

21. $P(-8, 6)$ $Q(-5, 5)$ $R(4, 2)$

B **22.** A wheelchair ramp is to be built at the town library. If the entrance to the library is 18 in. above ground, and the slope of the ramp is $\frac{1}{15}$, how far out from the building will the ramp start?

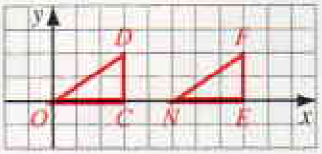
Complete.

23. A line with slope $\frac{3}{4}$ passes through points (2, 3) and (10, __?__).

24. A line with slope $-\frac{5}{2}$ passes through points (7, −4) and (__?__, 6).

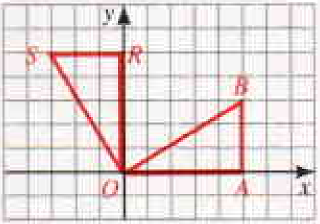
25. A line with slope m passes through points (p, q) and $(r,$ __?__$)$.

26. **a.** Find the slopes of $\overline{OD}$ and $\overline{NF}$.
b. Why is $\triangle OCD \cong \triangle NEF$?
c. Why is $\angle DOC \cong \angle FNE$?
d. Why is $\overline{OD} \parallel \overline{NF}$?
e. What do you think is true about the slopes of parallel lines?



Ex. 26

27. **a.** Show that $\triangle OAB \cong \triangle ORS$.
b. Why is $\overline{OB} \perp \overline{OS}$?
c. Find the product of the slopes of $\overline{OB}$ and $\overline{OS}$.



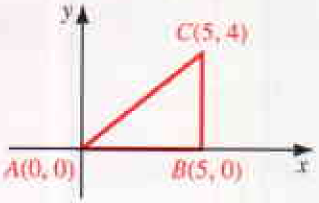
Ex. 27

In Exercises 28 and 29, (a) find the lengths of the sides of $\triangle RST$, (b) use the converse of the Pythagorean Theorem to show that $\triangle RST$ is a right triangle, and (c) find the product of the slopes of $\overline{RT}$ and $\overline{ST}$.

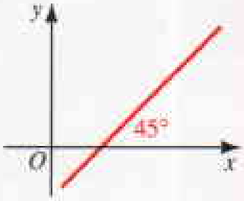
28. $R(4, 2)$, $S(-1, 7)$, $T(1, 1)$

29. $R(4, 3)$, $S(-3, 6)$, $T(2, 1)$

30. a. Show that $\tan \angle A$ = slope of $\overline{AC}$.
b. Use trigonometry to find $m\angle A$.



31. A line intersects the x-axis at a 45° angle. What is its slope?



C 32. A line passes through points $(-2, -1)$ and $(4, 3)$. Where does the line intersect the x-axis? the y-axis?

33. A line through $H(3, 1)$ and $J(5, a)$ has positive slope and makes a 60° angle measured counterclockwise with the positive x-axis. Find the value of a.

34. Find two values of k such that the points $(-3, 4)$, $(0, k)$, and $(k, 10)$ are collinear.

Algebra Review: *Exponents*

Rules of Exponents

When a and b are nonzero real numbers and m and n are integers:

	Examples
(1) $a^0 = 1$	$5^0 = 1$
(2) $a^m \cdot a^n = a^{m+n}$	$x^2 \cdot x^4 = x^{2+4} = x^6$
(3) $\frac{a^m}{a^n} = a^{m-n}$	$\frac{b^7}{b^3} = b^{7-3} = b^4$
(4) $(a^m)^n = a^{mn}$	$(y^3)^4 = y^{3\cdot 4} = y^{12}$
(5) $a^{-m} = \frac{1}{a^m}$	$6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

Simplify.

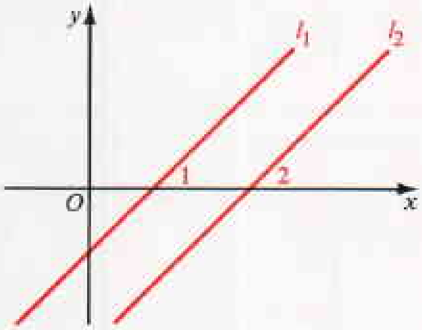
1. $(-6)^3$
2. $(-5)^4$
3. 3^{-2}
4. 2^{-3}
5. $(-4)^{-3}$
6. $\left(\frac{2}{3}\right)^{-2}$
7. $\left(\frac{5}{3}\right)^{-3}$
8. 15^0
9. $(-1)^{20}$
10. $(-1)^{99}$
11. $2^3 \cdot 2^2 \cdot 2^{-4}$
12. $4^2 \cdot 3^3 \cdot 2^{-3}$

Simplify. Use only positive exponents in your answers.

13. $r^5 \cdot r^8$
14. $x^{-1} \cdot x^{-2}$
15. $\frac{r^9}{r^4}$
16. $\frac{t^3}{t^5}$
17. $a \cdot a^{-1}$
18. $(x^2)^{-2}$
19. $(b^4)^2$
20. $(s^5)^3$
21. $(3y^2)(2y^4)$
22. $(4x^3y^2)(-3xy)$
23. $(5a^2b^3)(a^{-2}b)$
24. $(-2st^5)(-4st^{-3})$

13-3 *Parallel and Perpendicular Lines*

When you look at two parallel lines, you probably believe that the lines have equal slopes. This idea is illustrated by the photograph below. The parallel beams shown are needed to support a roof with a fixed pitch.



You can use trigonometry and properties of parallel lines to show the following for two nonvertical lines l_1 and l_2 (see the diagram at the right above):

1. $l_1 \parallel l_2$ if and only if $\angle 1 \cong \angle 2$
2. $\angle 1 \cong \angle 2$ if and only if $\tan \angle 1 = \tan \angle 2$
3. $\tan \angle 1 = \tan \angle 2$ if and only if slope of l_1 = slope of l_2

Therefore $l_1 \parallel l_2$ if and only if slope of l_1 = slope of l_2.

Although the diagram shows two lines with positive slope, this result can also be proved for two lines with negative slope. When the lines are parallel to the x-axis, both have slope zero.

Theorem 13-3

Two nonvertical lines are parallel if and only if their slopes are equal.

In Exercises 27–29 of the preceding section, you may have noticed that perpendicular lines, too, have slopes that are related in a special way. See Classroom Exercise 11 and Written Exercise 23 for proofs of the following theorem.

Theorem 13-4

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

$$m_1 \cdot m_2 = -1, \text{ or } m_1 = -\frac{1}{m_2}$$

Example Given points $S(5, -1)$ and $T(-3, 3)$, find the slope of every line (a) parallel to $\overleftrightarrow{ST}$ and (b) perpendicular to $\overleftrightarrow{ST}$.

Solution Slope of $\overleftrightarrow{ST} = \frac{3 - (-1)}{-3 - 5} = \frac{4}{-8} = -\frac{1}{2}$

a. Any line parallel to $\overleftrightarrow{ST}$ has slope $-\frac{1}{2}$. (Theorem 13-3)

b. Any line perpendicular to $\overleftrightarrow{ST}$ has slope

$-\frac{1}{-\frac{1}{2}} = -1 \cdot (-2) = 2.$ (Theorem 13-4)

Classroom Exercises

1. Given: $l \perp n$. Find the slope of line n if the slope of line l is:

a. 2 **b.** $\frac{4}{5}$ **c.** -4 **d.** not defined **e.** 0

The slopes of two lines are given. Are the lines parallel, perpendicular, or neither?

2. $\frac{3}{4}; \frac{12}{16}$ **3.** $1; -1$ **4.** $3; -3$ **5.** $-\frac{3}{4}; \frac{4}{3}$

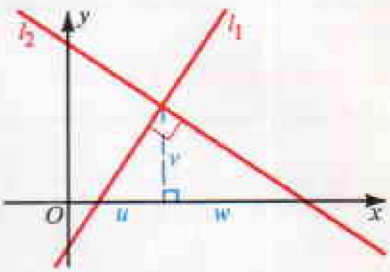
6. $3; \frac{-1}{3}$ **7.** $\frac{-2}{3}; \frac{2}{-3}$ **8.** $0; -1$ **9.** $\frac{5}{6}; \frac{6}{5}$

10. State two conditionals that are combined in the biconditional of Theorem 13-3.

11. The purpose of this exercise is to prove the statement: If two nonvertical lines are perpendicular, then the product of their slopes is -1. Supply the reason for each step.

Given: l_1 has slope m_1;
l_2 has slope m_2;
$l_1 \perp l_2$

Prove: $m_1 \cdot m_2 = -1$



Key steps of proof:

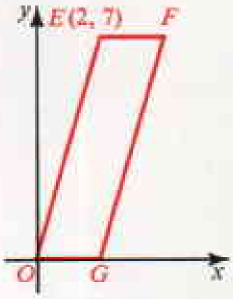
1. Draw the vertical segment shown.
2. $\frac{u}{v} = \frac{v}{w}$
3. $m_1 = \frac{v}{u}$
4. $m_2 = -\frac{v}{w}$
5. $m_1 \cdot m_2 = \left(\frac{v}{u}\right) \cdot \left(-\frac{v}{w}\right)$
6. $m_1 \cdot m_2 = \left(\frac{v}{u}\right) \cdot \left(-\frac{u}{v}\right) = -1$

Written Exercises

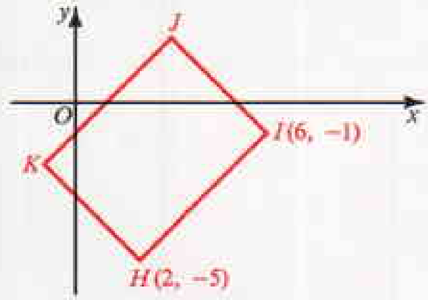
Find the slope of (a) $\overleftrightarrow{AB}$, (b) any line parallel to $\overleftrightarrow{AB}$, and (c) any line perpendicular to $\overleftrightarrow{AB}$.

A

1. $A(-2, 0)$ and $B(4, 4)$
2. $A(-3, 1)$ and $B(2, -1)$
3. In the diagram at the left below, $OEFG$ is a parallelogram. What is the slope of $\overline{OE}$? of $\overline{GF}$? of $\overline{OG}$? of $\overline{EF}$?



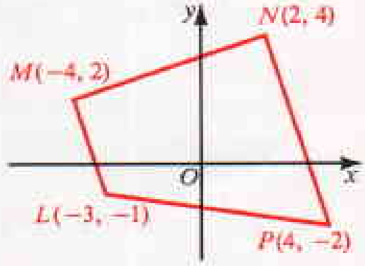
Ex. 3



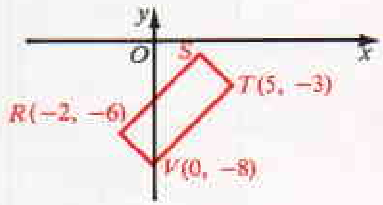
Ex. 4

4. In the diagram at the right above, $HIJK$ is a rectangle. What is the slope of $\overline{HI}$? of $\overline{JK}$? of $\overline{IJ}$? of $\overline{KH}$?

5. a. What is the slope of $\overline{LM}$? of $\overline{PN}$?
   b. Why is $\overline{LM} \parallel \overline{PN}$?
   c. What is the slope of $\overline{MN}$? of $\overline{LP}$?
   d. Why is $\overline{MN}$ not parallel to $\overline{LP}$?
   e. What special kind of quadrilateral is $LMNP$?



6. Quadrilateral $RSTV$ is known to be a parallelogram.
   a. What is the slope of $\overline{RV}$? of $\overline{TV}$?
   b. Why is $\overline{RV} \perp \overline{TV}$?
   c. Why is $\square RSTV$ a rectangle?
   d. Find the coordinates of S.



Find the slope of each side and each altitude of $\triangle ABC$.

7. $A(0, 0)$ $B(7, 3)$ $C(2, -5)$
8. $A(1, 4)$ $B(-1, -3)$ $C(4, -5)$

Use slopes to show that $\triangle RST$ is a right triangle.

9. $R(-3, -4)$ $S(2, 2)$ $T(14, -8)$
10. $R(-1, 1)$ $S(2, 4)$ $T(5, 1)$

B

11. Given the points $A(-6, -4)$, $B(4, 2)$, $C(6, 8)$, and $D(-4, 2)$ show that $ABCD$ is a parallelogram using two different methods.
    a. Show that opposite sides are parallel. b. Show that opposite sides are congruent.

12. Given: Points $E(-4, 1)$, $F(2, 3)$, $G(4, 9)$, and $H(-2, 7)$
 a. Show that $EFGH$ is a rhombus.
 b. Use slopes to verify that the diagonals are perpendicular.

13. Given: Points $R(-4, 5)$, $S(-1, 9)$, $T(7, 3)$ and $U(4, -1)$
 a. Show that $RSTU$ is a rectangle.
 b. Use the distance formula to verify that the diagonals are congruent.

14. Given: Points $N(-1, -5)$, $O(0, 0)$, $P(3, 2)$, and $Q(8, 1)$
 a. Show that $NOPQ$ is an isosceles trapezoid.
 b. Show that the diagonals are congruent.

Decide what special type of quadrilateral *HIJK* is. Then prove that your answer is correct.

15. $H(0, 0)$	$I(5, 0)$	$J(7, 9)$	$K(1, 9)$
16. $H(0, 1)$	$I(2, -3)$	$J(-2, -1)$	$K(-4, 3)$
17. $H(7, 5)$	$I(8, 3)$	$J(0, -1)$	$K(-1, 1)$
18. $H(-3, -3)$	$I(-5, -6)$	$J(4, -5)$	$K(6, -2)$

19. Point $N(3, -4)$ lies on the circle $x^2 + y^2 = 25$. What is the slope of the line that is tangent to the circle at N? (*Hint*: Recall Theorem 9-1.)

20. Point $P(6, 7)$ lies on the circle $(x + 2)^2 + (y - 1)^2 = 100$. What is the slope of the line that is tangent to the circle at P?

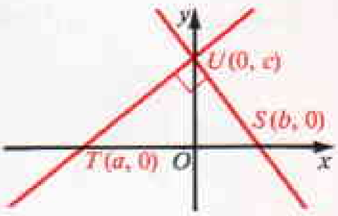
In Chapter 3 parallel lines are defined as coplanar lines that do not intersect. It is also possible to define parallel lines *algebraically* as follows:

Lines *a* and *b* are *parallel* if and only if slope of *a* = slope of *b* (or both *a* and *b* are vertical).

21. Use the algebraic definition to classify each statement as true or false.
 a. For any line l in a plane, $l \parallel l$.
 b. For any lines l and n in a plane, if $l \parallel n$, then $n \parallel l$.
 c. For any lines k, l, and n in a plane, if $k \parallel l$ and $l \parallel n$, then $k \parallel n$.

22. Refer to Exercise 21. Is parallelism of lines an equivalence relation? (See Exercise 15, page 43.) Explain.

C 23. This exercise shows another way to prove Theorem 13-4.
 a. Use the Pythagorean Theorem to prove:
 If $\overleftrightarrow{TU} \perp \overleftrightarrow{US}$, then the product of the slopes of $\overleftrightarrow{TU}$ and $\overleftrightarrow{US}$ equals -1. That is, prove $\left(-\frac{c}{a}\right) \cdot \left(-\frac{c}{b}\right) = -1$.



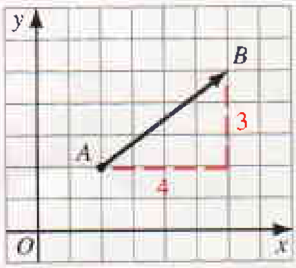
 b. Use the converse of the Pythagorean Theorem to prove:
 If $\left(-\frac{c}{a}\right) \cdot \left(-\frac{c}{b}\right) = -1$, then $\overleftrightarrow{TU} \perp \overleftrightarrow{US}$.

13-4 Vectors

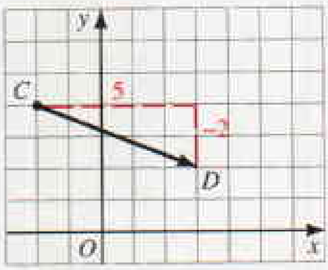
The journey of a boat or airplane can be described by giving its speed and direction, such as 50 km/h northeast. Any quantity such as force, velocity, or acceleration, that has both *magnitude* (size) and *direction*, is a **vector.**

When a boat moves from point A to point B, its journey can be represented by drawing an arrow from A to B, $\overrightarrow{AB}$ (read "*vector AB*"). If $\overrightarrow{AB}$ is drawn in the coordinate plane, then the journey can also be represented as an ordered pair.

$$\overrightarrow{AB} = (\text{change in } x, \text{ change in } y)$$



$\overrightarrow{AB} = (4, 3)$



$\overrightarrow{CD} = (5, -2)$

The **magnitude** of a vector $\overrightarrow{AB}$ is the length of the arrow from point A to point B and is denoted by the symbol $|\overrightarrow{AB}|$. You can use the Pythagorean Theorem or the Distance Formula to find the magnitude of a vector. In the diagrams above,

$$|\overrightarrow{AB}| = \sqrt{4^2 + 3^2} = 5$$

and $$|\overrightarrow{CD}| = \sqrt{5^2 + 2^2} = \sqrt{29}.$$

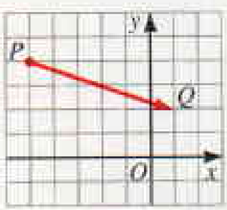
Example 1 Given: Points $P(-5, 4)$ and $Q(1, 2)$

a. Sketch $\overrightarrow{PQ}$.

b. Find $\overrightarrow{PQ}$.

c. Find $|\overrightarrow{PQ}|$.

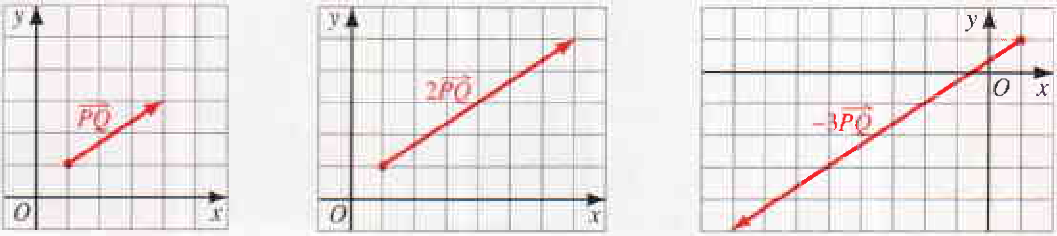
Solution **a.**



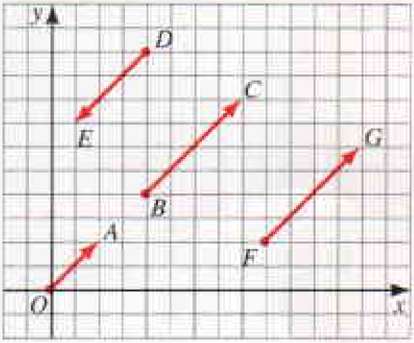
b. $\overrightarrow{PQ} = (1 - (-5), 2 - 4) = (6, -2)$

c. $|\overrightarrow{PQ}| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$

The symbol $2\overrightarrow{PQ}$ represents a vector that has twice the magnitude of $\overrightarrow{PQ}$ and has the same direction. If $\overrightarrow{PQ} = (3, 2)$, it should not surprise you that $2\overrightarrow{PQ} = (2 \cdot 3, 2 \cdot 2) = (6, 4)$. In general, if the vector $\overrightarrow{PQ} = (a, b)$, then $k\overrightarrow{PQ} = (ka, kb)$; $k\overrightarrow{PQ}$ is called a **scalar multiple** of $\overrightarrow{PQ}$. Multiplying a vector by a real number k multiplies the length of the vector by $|k|$. If $k < 0$, the direction of the vector reverses as well. This is illustrated in the diagrams below. What ordered pair represents $-3\overrightarrow{PQ}$?



Two vectors are *perpendicular* if the arrows representing them have perpendicular directions. Two vectors are *parallel* if the arrows representing them have the same direction or opposite directions. All the vectors shown at the right are parallel. Notice that $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are parallel even though the points O, A, B, and C are collinear.



Two vectors are **equal** if they have the same magnitude and the same direction. In the diagram, $\overrightarrow{BC} = \overrightarrow{FG}$.

You can tell by using slopes whether nonvertical vectors are parallel or perpendicular. Example 2 shows how.

Example 2 **a.** Show that $(9, -6)$ and $(-6, 4)$ are parallel.

b. Show that $(9, -6)$ and $(2, 3)$ are perpendicular.

Solution **a.** Slope of $(9, -6)$ is $\frac{-6}{9} = -\frac{2}{3}$.

Slope of $(-6, 4) = \frac{4}{-6} = -\frac{2}{3}$.

Since the slopes are equal, the vectors are parallel.

b. Slope of $(9, -6)$ is $\frac{-6}{9} = -\frac{2}{3}$.

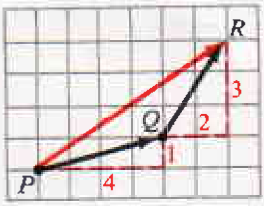
Slope of $(2, 3)$ is $\frac{3}{2}$.

Since $\frac{-2}{3} \cdot \frac{3}{2} = -1$, the vectors are perpendicular.

Vectors can be added by the following simple rule:

$$(a, b) + (c, d) = (a + c, b + d)$$

To see an application of adding vectors, suppose that a jet travels from P to Q and then from Q to R. The jet could have made the same journey by flying directly from P to R. $\overrightarrow{PR}$ is the **sum** of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. We abbreviate this fact by writing

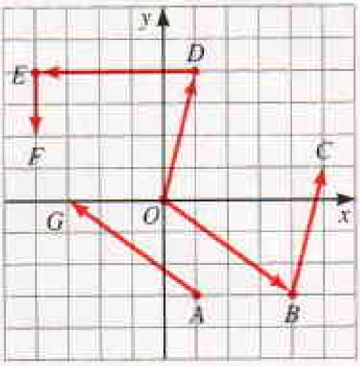


$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

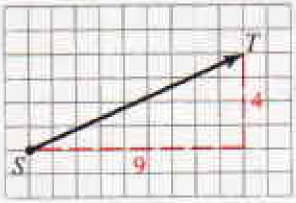
$$(4, 1) + (2, 3) = (6, 4)$$

Classroom Exercises

Exercises 1–4 refer to the figure at the right.



1. Name each vector as an ordered pair.
   a. $\overrightarrow{OB}$ **b.** $\overrightarrow{OD}$ **c.** $\overrightarrow{DE}$
   d. $\overrightarrow{EF}$ **e.** $\overrightarrow{BC}$ **f.** $\overrightarrow{AG}$
2. Find the magnitude of each vector in Exercise 1.
3. **a.** Is $\overrightarrow{BC}$ parallel to $\overrightarrow{OD}$? Explain.
   b. Is $\overrightarrow{BC} = \overrightarrow{OD}$? Explain.
   c. What kind of figure is $OBCD$? Explain.
4. **a.** Is $\overrightarrow{AG}$ parallel to $\overrightarrow{OB}$? Explain.
   b. Is $\overrightarrow{AG} = \overrightarrow{OB}$? Explain.



5. Refer to the diagram. Find $|\overrightarrow{ST}|$ and $\tan \angle S$.
6. Find each sum.
   a. $(3, 1) + (5, 6)$
   b. $(0, -6) + (7, 4)$
   c. $(-3, 10) + (-5, -12)$
7. Find each scalar multiple.
   a. $2(3, 1)$ **b.** $3(-5, 1)$ **c.** $-\frac{1}{2}(-6, 0)$
8. If $\overrightarrow{PQ}$ represents a wind blowing 45 km/h from the north, state two ways you could name the vector representing a wind blowing 45 km/h from the south.

Written Exercises

In Exercises 1–9 points A and B are given. Make a sketch. Then find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

A

1. $A(1, 1)$, $B(5, 4)$	**2.** $A(2, 0)$, $B(8, 8)$	**3.** $A(6, 1)$, $B(4, 3)$
4. $A(0, 5)$, $B(-3, 2)$	**5.** $A(3, 5)$, $B(-1, 7)$	**6.** $A(4, -2)$, $B(0, 0)$
7. $A(0, 0)$, $B(5, -9)$	**8.** $A(-3, 5)$, $B(3, 0)$	**9.** $A(-1, -1)$, $B(-4, -7)$

Use a grid and draw arrows to represent the following vectors. You can choose any starting point you like for each vector.

10. (3, 5) and 2(3, 5)
11. (4, −1) and 3(4, −1)
12. (−8, 4) and $\frac{1}{2}$(−8, 4)
13. (−6, −9) and $\frac{1}{3}$(−6, −9)
14. (4, 1) and −3(4, 1)
15. (6, −4) and $-\frac{1}{2}$(6, −4)

16. Name two vectors parallel to (3, −8).

17. The vectors (8, 6) and (12, k) are parallel. Find the value of k.

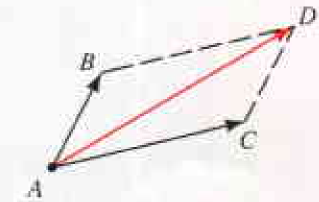
18. Show that (4, −5) and (15, 12) are perpendicular.

19. The vectors (8, k) and (9, 6) are perpendicular. Find the value of k.

Find each vector sum. Then illustrate each sum with a diagram like that on page 541.

20. (2, 1) + (3, 6)
21. (3, −5) + (4, 5)
22. (−8, 2) + (4, 6)
23. (−3, −3) + (7, 7)
24. (1, 4) + 2(3, 1)
25. (7, 2) +3(−1, 0)

B **26.** Two forces $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are pulling an object at point A. The single force $\overrightarrow{AD}$ that has the same effect as these two forces is their sum $\overrightarrow{AB} + \overrightarrow{AC}$. This sum can be found by completing parallelogram $ABDC$ as shown. Explain why the diagonal $\overrightarrow{AD}$ is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



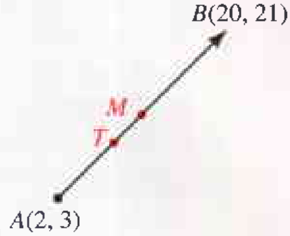
27. Make a drawing showing an object being pulled by the two forces $\overrightarrow{KX} = (-1, 5)$ and $\overrightarrow{KY} = (7, 3)$. What single force has the same effect as the two forces acting together? What is the magnitude of this force?

28. Repeat Exercise 27 for the forces $\overrightarrow{KX} = (2, -3)$ and $\overrightarrow{KY} = (-2, 3)$.

29. In the diagram, M is the midpoint of $\overline{AB}$ and T is a trisector point of $\overline{AB}$.

a. Complete: $\overrightarrow{AB}$ = (__?__, __?__), $\overrightarrow{AM}$ = (__?__, __?__) and $\overrightarrow{AT}$ = (__?__, __?__).

b. Find the coordinates of M and T.



30. Repeat Exercise 29 given the points $A(-10, 9)$ and $B(20, -15)$.

31. Use algebra to prove $|(ka, kb)| = |k| \cdot |(a, b)|$.

C **32.** **a.** Use definitions I and II below to prove that

$$k[(a, b) + (c, d)] = k(a, b) + k(c, d).$$

I. Definition of scalar multiple $\quad k(a, b) = (ka, kb)$
II. Definition of vector addition $\quad (a, b) + (c, d) = (a + c, b + d)$

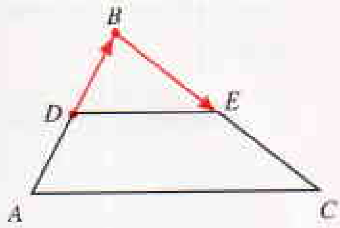
b. Make a diagram illustrating what you proved in part (a).

33. a. Given: $\overrightarrow{AB} = 2\overrightarrow{DB}$ and $\overrightarrow{BC} = 2\overrightarrow{BE}$

Supply the reasons for each step.

1. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
2. $\quad = 2\overrightarrow{DB} + 2\overrightarrow{BE}$
3. $\quad = 2(\overrightarrow{DB} + \overrightarrow{BE})$ (*Hint*: See Exercise 32.)
4. $\quad = 2\overrightarrow{DE}$

b. What theorem about midpoints does part (a) prove?



34. Suppose two nonvertical vectors (a, b) and (c, d) are perpendicular.

a. Use slopes to show that $\frac{bd}{ac} = -1$.

b. Show that $ac + bd = 0$.

c. The number $ac + bd$ is called the **dot product** of vectors (a, b) and (c, d). Complete: If (a, b) and (c, d) are perpendicular vectors, then their dot product __?__.

d. Verify the statement in part (c) for the vectors in Example 2(b) on page 540 and in Exercise 18 on page 542.

Mixed Review Exercises

1. On a number line, point A has coordinate -11 and B has coordinate 7. Find the coordinate of the midpoint of $\overline{AB}$.

2. If M is the midpoint of the hypotenuse $\overline{AB}$ of right triangle ABC, and $AM = 6$, find MB and MC.

3. The lengths of the bases of a trapezoid are 12 and 20. Find the length of the median.

4. If the length of one side of an equilateral triangle is $2a$, find the length of an altitude.

5. Find the measure of each interior angle of a regular hexagon.

6. Each side of a regular hexagon $ABCDEF$ has length x. Find AD and AC.

7. Find the measure of each exterior angle of a regular octagon.

8. Find the coordinates of the fourth vertex of a rectangle that has three vertices at $(-3, -2)$, $(2, -2)$, and $(2, 5)$.

9. The vertices of quad. $ABCD$ are $A(2, 0)$, $B(7, 0)$, $C(7, 5)$, and $D(2, 5)$. Find the area of quad. $ABCD$.

10. The vertices of $\triangle PQR$ are $P(0, 0)$, $Q(-6, 0)$, and $R(-6, 6)$. Find the area of $\triangle PQR$.

11. $\triangle DEF$ has vertices $D(-5, 1)$, $E(-2, -3)$, and $F(6, 3)$.

a. Use the distance formula to show that $\triangle DEF$ is a right triangle.

b. Use slopes to show that $\triangle DEF$ is a right triangle.

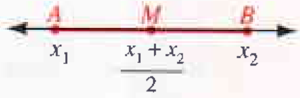
12. $\triangle ABC$ has vertices $A(6, 0)$, $B(4, 8)$, and $C(2, 6)$.

a. Find the slope of the altitude from B to $\overline{AC}$.

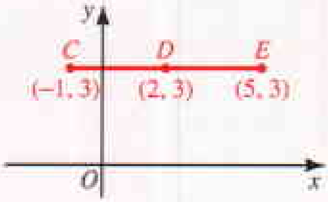
b. Find the slope of the perpendicular bisector of $\overline{AB}$.

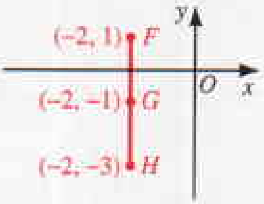
13-5 *The Midpoint Formula*

On a number line, if points A and B have coordinates x_1 and x_2, then the midpoint of $\overline{AB}$ has coordinate $\frac{x_1 + x_2}{2}$, the average of x_1 and x_2. (See Exercise 19 on page 47.)



This idea can be used to find the midpoint of any horizontal or vertical segment.



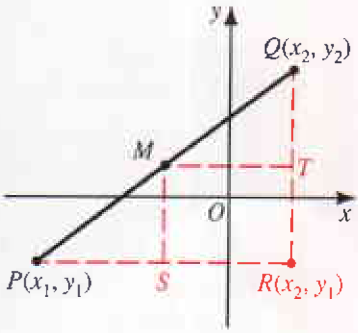


If a segment $\overline{PQ}$ is neither horizontal nor vertical, then the coordinates of its midpoint M can be found by drawing horizontal and vertical auxiliary lines as shown.

Since M is the midpoint of $\overline{PQ}$ and $\overline{MS} \parallel \overline{QR}$, S is the midpoint of $\overline{PR}$ (Theorem 5-10). Thus both S and M have x-coordinate $\frac{x_1 + x_2}{2}$.

Similarly, $\overline{MT} \parallel \overline{PR}$, so T is the midpoint of $\overline{QR}$. Thus both T and M have y-coordinate $\frac{y_1 + y_2}{2}$.

This discussion leads to the following theorem.



Theorem 13-5 *The Midpoint Formula*

The midpoint of the segment that joins points (x_1, y_1) and (x_2, y_2) is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 1 Find the midpoint of the segment that joins $(-11, 3)$ and $(8, -7)$.

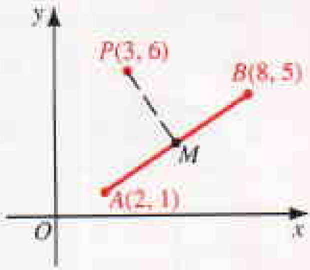
Solution The x-coordinate of the midpoint is

$$\frac{x_1 + x_2}{2} = \frac{-11 + 8}{2} = \frac{-3}{2}, \text{ or } -\frac{3}{2}.$$

The y-coordinate of the midpoint is

$$\frac{y_1 + y_2}{2} = \frac{3 - 7}{2} = \frac{-4}{2} = -2.$$

The midpoint is $\left(-\frac{3}{2}, -2\right)$.



Example 2 Given points $A(2, 1)$ and $B(8, 5)$, show that $P(3, 6)$ is on the perpendicular bisector of $\overline{AB}$.

Solution 1 Join P to M, the midpoint of $\overline{AB}$ and show that $\overline{PM} \perp \overline{AB}$.

Step 1 $M = \left(\frac{2 + 8}{2}, \frac{1 + 5}{2}\right) = (5, 3)$

Step 2 Slope of $\overline{AB} = \frac{5 - 1}{8 - 2} = \frac{4}{6} = \frac{2}{3}$

Slope of $\overline{PM} = \frac{3 - 6}{5 - 3} = \frac{-3}{2}$

Step 3 Since the product of the slopes of $\overline{AB}$ and $\overline{PM}$ is -1, $\overline{PM} \perp \overline{AB}$.

Solution 2 Show that P is equidistant from A and B and apply Theorem 4-6, page 153.

Step 1 $PA = \sqrt{(3 - 2)^2 + (6 - 1)^2} = \sqrt{26}$

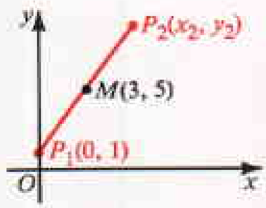
$PB = \sqrt{(3 - 8)^2 + (6 - 5)^2} = \sqrt{26}$

Step 2 Since $PA = PB$, P is on the perpendicular bisector of $\overline{AB}$.

Classroom Exercises

Find the coordinates of the midpoint of the segment that joins the given points.

1. (3, 5) and (7, 5)
2. (0, 4) and (4, 3)
3. (−2, 2) and (6, 4)
4. (−3, 7) and (−7, −5)
5. (−1, −3) and (−3, 6)
6. (2b, 3) and (4, −5)
7. (t, 2) and (t + 4, −4)
8. (a, n) and (d, p)



9. $M(3, 5)$ is the midpoint of $\overline{P_1P_2}$, where P_1 has coordinates (0, 1). Find the coordinates of P_2.
10. Point (1, −1) is the midpoint of $\overline{AB}$, where A has coordinates (−1, 3). Find the coordinates of B.

Written Exercises

Find the coordinates of the midpoint of the segment that joins the given points.

A

1. (0, 2) and (6, 4)
2. (−2, 6) and (4, 3)
3. (6, −7) and (−6, 3)
4. (a, 4) and (a + 2, 0)
5. (2.3, 3.7) and (1.5, −2.9)
6. (a, b) and (c, d)

Find the length, slope, and midpoint of $\overline{PQ}$.

7. $P(3, -8)$, $Q(-5, 2)$ **8.** $P(-3, 4)$, $Q(7, 8)$ **9.** $P(-7, 11)$, $Q(1, -4)$

In Exercises 10–12, M is the midpoint of $\overline{AB}$, where the coordinates of A are given. Find the coordinates of B.

10. $A(4, -2)$; $M(4, 4)$ **11.** $A(1, -3)$; $M(5, 1)$ **12.** $A(r, s)$; $M(0, 2)$

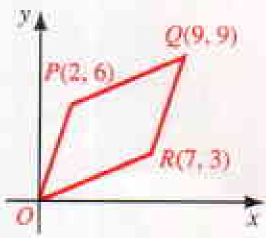
B **13.** Given points $A(0, 0)$ and $B(8, 4)$, show that $P(2, 6)$ is on the perpendicular bisector of $\overline{AB}$ by using both of the methods in Example 2.

14. **a.** Given points $R(1, 0)$, $S(7, 4)$, and $T(11, -2)$, show that $\triangle RST$ is isosceles.
b. The altitude from the vertex meets the base at K. Find the coordinates of K.

15. Find the midpoints of the legs, then the length of the median of the trapezoid with vertices $C(-4, -3)$, $D(-1, 4)$, $E(4, 4)$, and $F(7, -3)$.

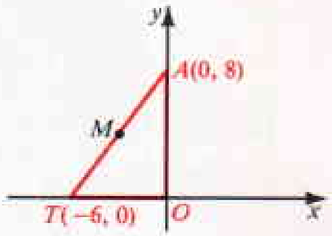
16. Find the length of the longest median of the triangle with vertices $X(-2, 3)$, $Y(6, -3)$, and $Z(4, 7)$.

17. **a.** Verify that $\overline{OQ}$ and $\overline{PR}$ have the same midpoint.
b. Part (a) shows that the diagonals of $OPQR$ bisect each other. Therefore $OPQR$ is a __?__.
c. Use slopes to verify that the opposite sides of $OPQR$ are parallel.
d. Use the distance formula to verify that the opposite sides are congruent.



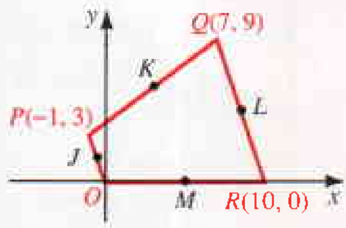
18. Graph the points $A(-5, 0)$, $B(3, 2)$, $C(5, 6)$, and $D(-3, 4)$. Then show that $ABCD$ is a parallelogram by two different methods.
a. Show that one pair of opposite sides are both congruent and parallel.
b. Show that the diagonals bisect each other (have the same midpoint).

19. In right $\triangle OAT$, M is the midpoint of $\overline{AT}$.
a. M has coordinates (__?__, __?__).
b. Find, and compare, the lengths MA, MT, and MO.
c. State a theorem from Chapter 5 suggested by this exercise.
d. Find an equation of the circle that circumscribes $\triangle OAT$.



20. Given points $A(1, 1)$, $B(13, 9)$, and $C(3, 7)$. D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{AC}$.
a. Find the coordinates of D and E.
b. Use slopes to show that $\overline{DE} \parallel \overline{BC}$.
c. Use the distance formula to show that $DE = \frac{1}{2}BC$.

21. **a.** Find the coordinates of the midpoints J, K, L, and M.
b. What kind of figure is $JKLM$? Prove it.



Ex. 21

22. Suppose E is on $\overline{PQ}$ and $PE = \frac{1}{4}PQ$. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, where $x_1 < x_2$, show that $E = \left(\frac{3}{4}x_1 + \frac{1}{4}x_2, \frac{3}{4}y_1 + \frac{1}{4}y_2\right)$.

C **23.** Suppose F is on $\overline{PQ}$ and $PF = \frac{3}{8}PQ$. If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, where $x_1 < x_2$, find the coordinates of F. (*Hint*: See Exercise 22.)

24. Given points $P(2, 1)$ and $D(7, 11)$, find the coordinates of a point T on $\overline{PD}$ such that $\frac{PT}{TD} = \frac{2}{3}$.

Self-Test 1

For each pair of points find (a) the distance between the two points and (b) the midpoint of the segment that joins the two points.

1. $(5, 1)$ and $(3, 1)$

2. $(8, -6)$ and $(0, 0)$

3. $(-2, 7)$ and $(8, -3)$

4. $(-3, 2)$ and $(-5, 7)$

Write an equation of the circle described.

5. Center at the origin; radius 9

6. Center $(-1, 2)$; radius 5

7. Find the center and the radius of the circle $(x + 2)^2 + (y - 3)^2 = 36$.

Find the slope of the line through the points named.

8. $(0, 0)$ and $(7, 4)$

9. $(-4, 2)$ and $(1, -1)$

10. For which is slope *not* defined, a horizontal line or a vertical line?

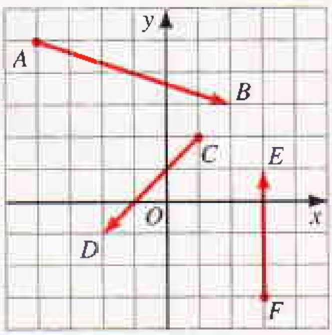
11. Given $P(3, -2)$ and $Q(5, 2)$, find:
a. the slope of any line parallel to $\overleftrightarrow{PQ}$
b. the slope of any line perpendicular to $\overleftrightarrow{PQ}$

12. Name each vector as an ordered pair.
a. $\overrightarrow{AB}$ **b.** $\overrightarrow{CD}$ **c.** $\overrightarrow{FE}$

13. Find the magnitude of each vector in Exercise 12.

14. Complete.
a. $(-3, 2) + (7, -11) =$ _?_
b. $3(4, -1) + (-2)(-5, 3) =$ _?_

15. If $M(-3, 7)$ is the midpoint of $\overline{PQ}$, where P has coordinates $(9, -4)$, find the coordinates of Q.



Exs. 12, 13

Lines and Coordinate Geometry Proofs

Objectives

1. Identify the slope and y-intercept of the line specified by a given equation.
2. Draw the graph of the line specified by a given equation.
3. Write an equation of a line when given either one point and the slope of the line, or two points on the line.
4. Determine the intersection of two lines.
5. Given a polygon, choose a convenient placement of coordinate axes and assign appropriate coordinates.
6. Prove statements by using coordinate geometry methods.

13-6 *Graphing Linear Equations*

A **linear equation** is an equation whose graph is a line. As you will learn in this section and the next, linear equations can be written in different forms: *standard form*, *slope-intercept form*, and *point-slope form*. We state a theorem for the standard form, but omit the proof.

Theorem 13-6 *Standard Form*

The graph of any equation that can be written in the form

$$Ax + By = C$$

where A and B are not both zero, is a line.

The advantage of the standard form is that it is easy to determine the points where the line crosses the x-axis and the y-axis. If a line intersects the x-axis at the point $(a, 0)$, then its *x-intercept* is a; if it intersects the y-axis at the point $(0, b)$, then its *y-intercept* is b.

Example 1 Graph the line $2x - 3y = 12$.

Solution Since two points determine a line, begin by plotting two convenient points, such as the points where the line crosses the axes. Then draw the line through the points.

To find the x-intercept, let $y = 0$.

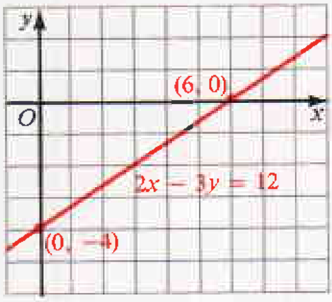
$$2x - 3(0) = 12$$
$$x = 6$$

Thus $(6, 0)$ is a point on the line.

To find the y-intercept, let $x = 0$.

$$2(0) - 3y = 12$$
$$y = -4$$

Thus $(0, -4)$ is a point on the line.

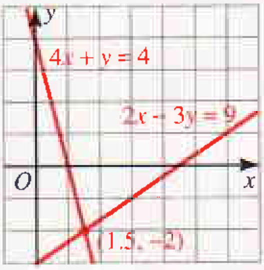


Example 2 Use algebra to find the intersection of the lines $2x - 3y = 9$ and $4x + y = 4$. Illustrate by drawing the graphs of the two lines.

Solution

$$\begin{aligned} 2x - 3y &= 9 && \text{(First equation)} \\ 12x + 3y &= 12 && \text{(Second equation} \times 3) \\ \hline 14x &= 21 && \text{(Add to eliminate } y.) \\ x &= 1.5 \\ 4(1.5) + y &= 4 && \text{(Substitution)} \\ y &= -2 \end{aligned}$$

The point of intersection is $(1.5, -2)$.



The equations in Examples 1 and 2 are all written in standard form. These equations can also be written in the *slope-intercept form* $y = mx + b$. This form tells you at a glance what the line's slope and y-intercept are.

standard form	*slope-intercept form*	*slope*	*y-intercept*
$2x - 3y = 12$	$y = \frac{2}{3}x - 4$	$\frac{2}{3}$	-4
$2x - 3y = 9$	$y = \frac{2}{3}x - 3$	$\frac{2}{3}$	-3
$4x + y = 4$	$y = -4x + 4$	-4	4

Theorem 13-7 *Slope-Intercept Form*

A line with the equation $y = mx + b$ has slope m and y-intercept b.

Proof:

When $x = 0$, $y = b$. So b is the y-intercept.

When $x = 1$, $y = m + b$.
Let $(x_1, y_1) = (0, b)$ and $(x_2, y_2) = (1, m + b)$.

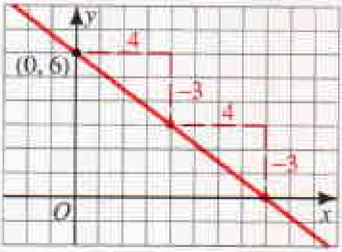
Then the slope is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(m + b) - b}{1 - 0} = m$.

Example 3 Graph the line $y = -\frac{3}{4}x + 6$.

Solution The slope is $\frac{-3}{4}$ and the y-intercept is 6.

Step 1 Start at the point (0, 6).

Step 2 Use $\frac{\text{change in y}}{\text{change in x}} = \frac{-3}{4}$ to find other points of the line. (See Example 2, page 530.)



Classroom Exercises

1. Which points lie on the line $3x - 2y = 12$?
   a. (0, 4) **b.** (2, −3) **c.** $\left(3, \frac{3}{2}\right)$ **d.** (0, −6)
2. Which point is the intersection of $x + 2y = 8$ and $2x + 3y = 10$?
   a. (−2, 5) **b.** (−4, 6) **c.** (2, 3) **d.** (−1, 4)

Find the x- and y-intercepts of each line.

3. $2x + 3y = 6$
4. $3x - 5y = 15$
5. $-4x + 3y = 24$
6. $x + 3y = 9$
7. $y = 5x - 10$
8. $y = 2x + 5$

Find the slope and y-intercept of each line.

9. $y = \frac{2}{5}x - 9$
10. $2x + y = 8$
11. $3x - 4y = 6$
12. What is the slope of the line $y = 4$? Name three points that lie on the line.
13. The graph of $x = 5$ is a vertical line through (5, 0). Name three other points on the line and check to see if their coordinates satisfy the equation.

Written Exercises

A

1. On the same axes, graph $y = mx$ for $m = 2, -2, \frac{1}{2}$, and $-\frac{1}{2}$.
2. On the same axes, graph $y = mx + 2$ for $m = 3, -3, \frac{1}{3}$, and $-\frac{1}{3}$.
3. On the same axes, graph $y = \frac{1}{2}x + b$ for $b = 0, 2, 4, -2$, and -4.
4. On the same axes, graph $y = -\frac{2}{3}x + b$ for $b = 0, 3, 6, -3$, and -6.
5. On the same axes, graph the lines $y = 0$, $y = 3$, and $y = -3$.
6. On the same axes, graph the lines $x = 0$, $x = 2$, and $x = -2$.

Find the x-intercept and y-intercept of each line. Then graph the equation.

7. $3x + y = -21$ **8.** $4x - 5y = 20$ **9.** $3x + 2y = 12$

10. $3x - 2y = 12$ **11.** $5x + 8y = 20$ **12.** $3x + 4y = -18$

Find the slope and y-intercept of each line. Plot the y-intercept. Then, using the slope, plot one more point. Finally, graph the line.

13. $y = 2x - 3$ **14.** $y = 2x + 3$ **15.** $y = -4x$

16. $y = \frac{3}{4}x + 1$ **17.** $y = -\frac{2}{3}x - 4$ **18.** $y = \frac{5}{3}x - 2$

Find the slope and y-intercept of each line.

Example $x + 3y = -6$

Solution Write the equation in slope-intercept form.

$$3y = -x - 6$$

$$y = -\frac{1}{3}x - 2$$

The slope is $-\frac{1}{3}$. The y-intercept is -2.

19. $4x + y = 10$ **20.** $2x - y = 5$ **21.** $5x - 2y = 10$

22. $3x + 4y = 12$ **23.** $x - 4y = 6$ **24.** $4x + 3y = 8$

Solve each pair of equations algebraically. Then draw the graphs of the equations and label their intersection point.

25. $x + y = 3$, $x - y = -1$

26. $2x + y = 7$, $3x + y = 9$

27. $x + 2y = 10$, $3x - 2y = 6$

28. $3x + 2y = -30$, $y = x$

29. $4x + 5y = -7$, $2x - 3y = 13$

30. $3x + 2y = 8$, $-x + 3y = 12$

B

31. a. Find the slopes of the lines $6x + 3y = 10$ and $y = -2x + 5$.
b. Do the lines intersect?
c. What happens when you solve these equations algebraically?

32. Give a geometric reason and an algebraic reason why the lines $y = 3x - 5$ and $y = 3x + 5$ do not intersect.

33. a. Find the slopes of the lines $2x - y = 7$ and $x + 2y = 4$.
b. What can you conclude about the lines? State the theorem that supports your answer.

34. a. On the same axes, graph

$$y = -2,\ x = -3, \text{ and } 2x + 3y = 6.$$

b. Find the coordinates of the three points where the lines intersect.
c. Find the area of the triangle determined by the three lines.

35. a. On the same axes, graph

$$y = \frac{1}{2}x - 2, \quad y = -2x + 3, \quad \text{and} \quad y = 3x + 8.$$

b. Find the coordinates of the three points where the lines intersect.

c. Find the area of the triangle determined by the three lines.

36. Find the area of the region inside the circle $x^2 + y^2 = 2$ *and* above the line $y = 1$.

C **37.** Use algebra to find each point at which the line $x - 2y = -5$ intersects the circle $x^2 + y^2 = 25$. Graph both equations to verify your answer.

38. a. Verify that the point $P(4, -2)$ is on the line $2x - y = 10$ and on the circle $x^2 + y^2 = 20$.

b. Show that the segment joining the center of the circle to P is perpendicular to the line.

c. What do parts (a) and (b) tell you about the line and the circle?

39. Graph each equation.

a. $|x| = |y|$ **b.** $|x| + |y| = 6$ **c.** $|x| + 2|y| = 4$

Explorations

These exploratory exercises can be done using a graphing calculator.

Graph the lines $y = 2x$, $y = 2x + 1$, and $y = 2x + 3$ on the same screen. What do you notice about these lines?

What theorem does this illustrate?

Use what you have observed to write an equation of the line whose y-intercept is 7 and that is parallel to $y = 2x$.

Graph the lines $y = 2x$ and $y = -\frac{1}{2}x$ on the same screen.

Graph the lines $y = \frac{2}{3}x$ and $y = -\frac{3}{2}x$ on the same screen.

What do you notice about both pairs of lines?

What theorem does this illustrate?

Use what you have observed to write an equation of the line through the origin that is perpendicular to $y = \frac{4}{5}x$.

Challenge

Draw segments that divide an obtuse triangle into acute triangles.

13-7 *Writing Linear Equations*

In the previous section, you were given a linear equation and asked to draw its graph. In this section you will be given information about a graph and asked to find an equation of the line described.

Example 1 Find an equation of each line described.

a. Slope $= -\frac{5}{3}$, y-intercept $= 4$

b. x-intercept $= -6$, y-intercept $= 3$

Solution **a.** $y = mx + b$

$y = -\frac{5}{3}x + 4$

b. Because the y-intercept is 3, you have $b = 3$.
Because the points $(-6, 0)$ and $(0, 3)$ lie on the line,

$$\text{slope} = \frac{3 - 0}{0 - (-6)} = \frac{3}{6} = \frac{1}{2}.$$

Since the slope is $\frac{1}{2}$, you have $m = \frac{1}{2}$.
Now substitute into the equation $y = mx + b$ to get

$$y = \frac{1}{2}x + 3.$$

Both linear equations in Example 1 were written in slope-intercept form. This form is very easy to use if the y-intercept is given. If the y-intercept is not given, the *point-slope form* can be used.

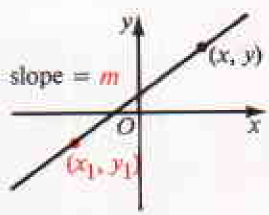
Theorem 13-8 *Point-Slope Form*

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1).$$

Proof:

Let (x, y) be any point on the line. Since the line also contains the point (x_1, y_1) the slope must, by definition, equal $\frac{y - y_1}{x - x_1}$.



From $m = \frac{y - y_1}{x - x_1}$,

we get $y - y_1 = m(x - x_1)$.

Example 2 Find an equation of each line described.

a. The line through (1, 2) and parallel to the line $y = 3x - 7$
b. The line through (1, 2) and perpendicular to the line $y = 3x - 7$
c. The line through the points $(-3, 0)$ and (1, 8)

Solution

a. If the line is parallel to the line $y = 3x - 7$, its slope must be 3. Substituting in $y - y_1 = m(x - x_1)$ gives $y - 2 = 3(x - 1)$, or $y = 3x - 1$.

b. The required line has slope $-\frac{1}{3}$. (Why?) Thus an equation in point-slope form is $y - 2 = -\frac{1}{3}(x - 1)$, or $y = -\frac{1}{3}x + \frac{7}{3}$, or $x + 3y = 7$.

c. First find the slope: $m = \dfrac{8 - 0}{1 - (-3)} = 2$

Then use the point-slope form with *either* given point.
Using $(-3, 0)$, the equation is $y - 0 = 2[x - (-3)]$, or $y = 2x + 6$.
Using (1, 8), the equation is $y - 8 = 2(x - 1)$, or $y = 2x + 6$.

Classroom Exercises

Give an equation of each line described.

1. Slope $= -\frac{1}{2}$; y-intercept $= 5$

2. Slope $= \frac{3}{7}$; y-intercept $= 8$

3. x-intercept $= 2$; y-intercept $= 4$

4. x-intercept $= 2$; y-intercept $= -6$

5. The x-axis

6. The y-axis

7. y-intercept $= -3$; parallel to $y = -\frac{4}{5}x + 2$

8. y-intercept $= 0$; perpendicular to $y = -\frac{7}{4}x + 9$

9. Slope $= \frac{5}{8}$; passes through (3, 4)

10. Slope $= -2$; passes through (8, 6)

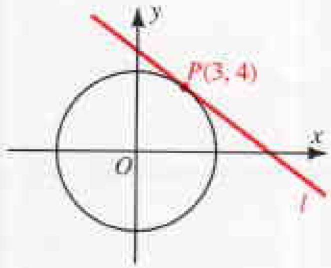
State the slope of the line and name two points on the line.

11. $y = -(x + 7)$

12. $y + 2 = \frac{1}{2}(x - 5)$

13. $y - c = \frac{a}{b}(x - d)$

14. Line l is tangent to $\odot O$ at point $P(3, 4)$.
a. Find the radius of the circle.
b. Give an equation of the circle.
c. Find the slope of line l.
d. Give an equation of line l.



Written Exercises

Give an equation of each line described. Use the form specified by your teacher.

A

	1.	2.	3.	4.	5.	6.
slope	2	-3	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{7}{5}$	$-\frac{3}{2}$
y-intercept	5	6	-8	-9	8	-7

	7.	8.	9.	10.
x-intercept	8	9	-8	-5
y-intercept	2	-3	4	-2

	11.	12.	13.	14.	15.	16.
point	(1, 2)	(3, 8)	$(-3, 5)$	$(6, -6)$	$(-4, 0)$	$(-10, 3)$
slope	5	4	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{2}{5}$

17. line through (1, 1) and (4, 7)

18. line through $(-1, -3)$ and (2, 1)

19. line through $(-3, 1)$ and (3, 3)

20. line through $(-2, -1)$ and $(-6, -5)$

21. vertical line through $(2, -5)$

22. horizontal line through (3, 1)

23. line through $(5, -3)$ and parallel to the line $x = 4$

24. line through $(-8, -2)$ and parallel to the line $x = 5$

B **25.** line through (5, 7) and parallel to the line $y = 3x - 4$

26. line through $(-1, 3)$ and parallel to the line $3x + 5y = 15$

27. line through $(-3, -2)$ and perpendicular to the line $8x - 5y = 0$

28. line through (8, 0) and perpendicular to the line $3x + 4y = 12$

29. perpendicular bisector of the segment joining (0, 0) and (10, 6)

30. perpendicular bisector of the segment joining $(-3, 7)$ and (5, 1)

31. the line through (5, 5) that makes a 45° angle measured counterclockwise from the positive x-axis

32. the line through the origin that makes a 135° angle measured counterclockwise from the positive x-axis

33. Find each value of k for which the lines $y = 9kx - 1$ and $kx + 4y = 12$ are perpendicular.

34. Quad. $BECK$ is known to be a rhombus. Two of the vertices are $B(3, 5)$ and $C(7, -3)$.

a. Find the slope of diagonal $\overline{EK}$.

b. Find an equation of $\overleftrightarrow{EK}$.

35. Find the center of the circle that passes through (2, 10), (10, 6), and (−6, −6).

Exercises 36–39 refer to $\triangle QRS$ with vertices $Q(-6, 0)$, $R(12, 0)$, and $S(0, 12)$.

C 36. a. Find the equations of the three lines that contain the medians.
b. Show that the three medians meet in a point G (called the *centroid*). (*Hint*: Solve two equations simultaneously and show that their solution satisfies the third equation.)
c. Show that the length QG is $\frac{2}{3}$ of the length of the median from Q.

37. a. Find the equations of the three perpendicular bisectors of the sides of $\triangle QRS$.
b. Show that the three perpendicular bisectors meet in a point C (called the *circumcenter*). (See the hint from Exercise 36(b).)
c. Show that C is equidistant from Q, R, and S by using the distance formula.
d. Find the equation of the circle that can be circumscribed about $\triangle QRS$.

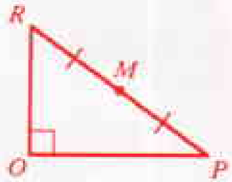
38. a. Find the equations of the three lines that contain the altitudes of $\triangle QRS$.
b. Show that the three altitudes meet in a point H (called the *orthocenter*).

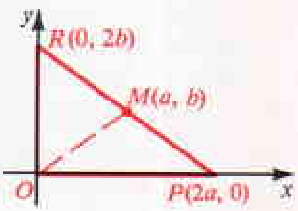
39. a. Refer to Exercises 36, 37, and 38. Use slopes to show that the points C, G, and H are collinear. (The line through these points is called *Euler's Line*.)
b. Show that $GH = 2GC$.

13-8 *Organizing Coordinate Proofs*

We will illustrate coordinate geometry methods by proving Theorem 5-15:

The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.





Proof:

Let $\overleftrightarrow{OP}$ and $\overleftrightarrow{OR}$ be the x-axis and y-axis.
Let P and R have the coordinates shown.

Then the coordinates of M are (a, b).

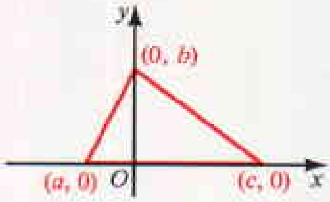
$MO = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$
$MP = \sqrt{(a - 2a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$
Thus $MO = MP$.

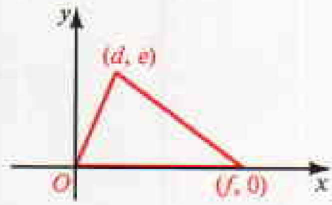
By the definition of midpoint, $MP = MR$.

Hence $MO = MP = MR$.

Notice that $2a$ and $2b$ are convenient choices for coordinates since they lead to expressions that do not contain fractions for the coordinates of M.

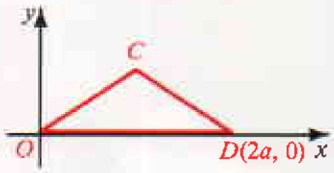
If you have a right triangle, such as $\triangle POR$ on page 556, the most convenient place to put the x-axis and y-axis is usually along the legs of the triangle. If a triangle is not a right triangle, the two most convenient ways to place your axes are shown below. Notice that these locations for the axes maximize the number of times zero is a coordinate of a vertex.



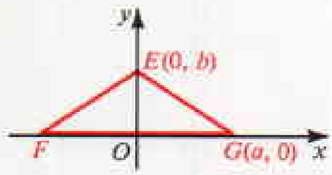


Some common ways of placing coordinate axes on other special figures are shown below.

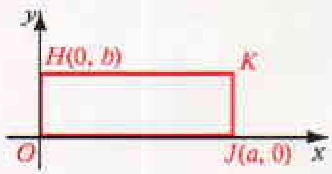
$\triangle COD$ is isosceles; $CO = CD$.
Then C can be labeled (a, b).



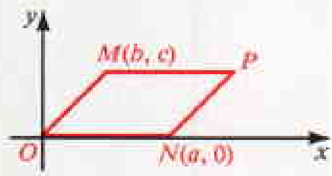
$\triangle EFG$ is isosceles; $EF = EG$.
Then F can be labeled $(-a, 0)$.



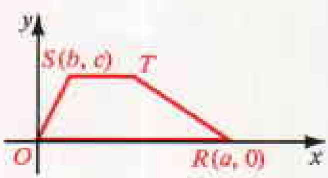
$HOJK$ is a rectangle.
Then K can be labeled (a, b).



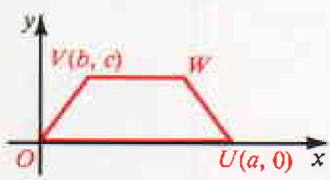
$MONP$ is a parallelogram.
Then P can be labeled $(a + b, c)$.



$ROST$ is a trapezoid.
Then T can be labeled (d, c).



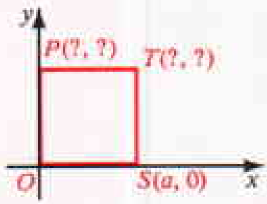
$UOVW$ is an isosceles trapezoid.
Then W can be labeled $(a - b, c)$.



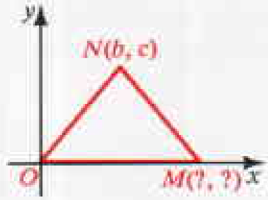
Classroom Exercises

Supply the missing coordinates without introducing any new letters.

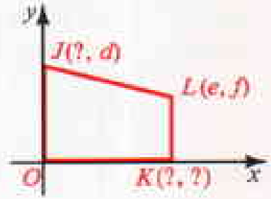
1. *POST* is a square.



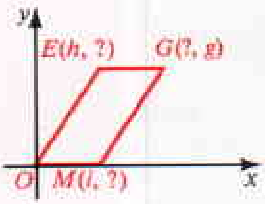
2. △*MON* is isosceles.



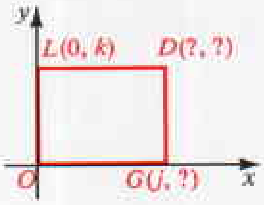
3. *JOKL* is a trapezoid.



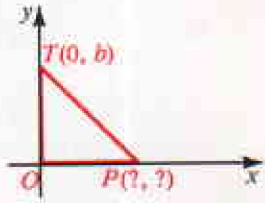
4. *GEOM* is a parallelogram.



5. *GOLD* is a rectangle.



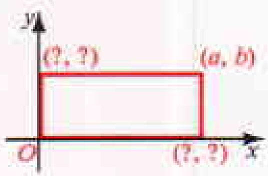
6. Rt. △*TOP* is isosceles.



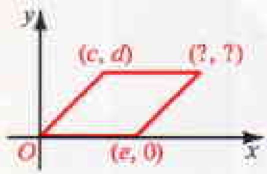
Written Exercises

Copy the figure. Supply the missing coordinates without introducing any new letters.

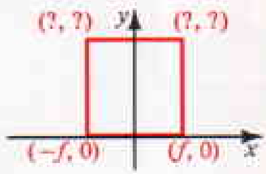
A 1. Rectangle



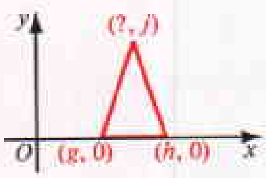
2. Parallelogram



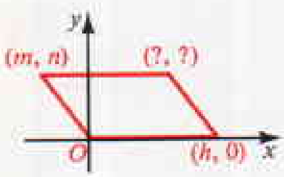
3. Square



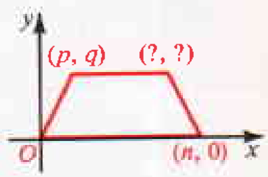
4. Isosceles triangle



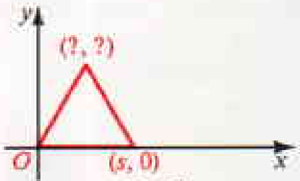
5. Parallelogram



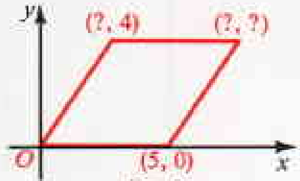
6. Isosceles trapezoid



B 7. An equilateral triangle is shown below. Express the missing coordinates in terms of s.



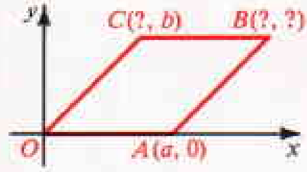
Ex. 7



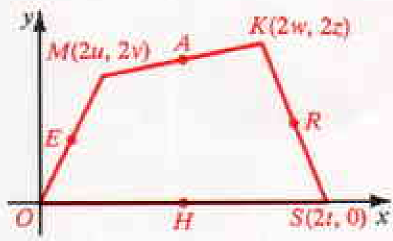
Ex. 8

8. A rhombus is shown above. Find the missing coordinates.

9. Rhombus $OABC$ is shown at the right. Express the missing coordinates in terms of a and b. (*Hint*: See Exercise 8.)



10. Supply the missing coordinates to prove: The segments that join the midpoints of opposite sides of any quadrilateral bisect each other. Let H, E, A, and R be the midpoints of the sides of quadrilateral $SOMK$. Choose axes and coordinates as shown.



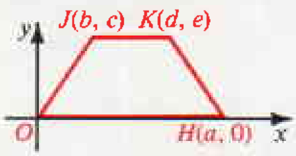
- **a.** R has coordinates (__?__, __?__).
- **b.** E has coordinates (__?__, __?__).
- **c.** The midpoint of $\overline{RE}$ has coordinates (__?__, __?__).
- **d.** A has coordinates (__?__, __?__).
- **e.** H has coordinates (__?__, __?__).
- **f.** The midpoint of $\overline{AH}$ has coordinates (__?__, __?__).
- **g.** Because (__?__, __?__) is the midpoint of both $\overline{RE}$ and $\overline{AH}$, $\overline{RE}$ and $\overline{AH}$ bisect each other.

Draw the figure named. Select axes and label the coordinates of the vertices in terms of a single letter.

C 11. a regular hexagon

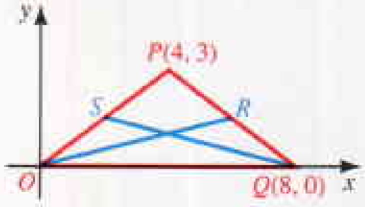
12. a regular octagon

13. Given isosceles trapezoid $HOJK$ and the axes and coordinates shown, use the definition of an isosceles trapezoid to prove that $e = c$ and $d = a - b$.



13-9 *Coordinate Geometry Proofs*

It is easy to verify that $\triangle OPQ$ is an isosceles triangle. Knowing this, we can deduce that medians $\overline{OR}$ and $\overline{QS}$ are congruent by using the midpoint and distance formulas.



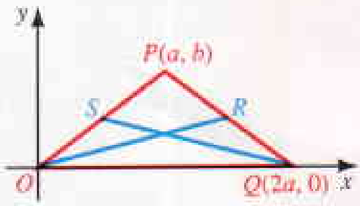
In order to give a coordinate proof that the medians to the legs are congruent for *any* isosceles triangle, and not just for the specific isosceles triangle above, you could use the figure below. Compare the general coordinates given in the figure below with the specific coordinates given for the triangle above. A coordinate proof follows.

Example 1

Prove that the medians to the legs of an isosceles triangle are congruent.

Proof:

Let OPQ be any isosceles triangle with $PO = PQ$. Choose convenient axes and coordinates as shown.



By the midpoint formula,

S has coordinates $\left(\frac{a}{2}, \frac{b}{2}\right)$ and R has coordinates $\left(\frac{3a}{2}, \frac{b}{2}\right)$.

By the distance formula,

$$OR = \sqrt{\left(\frac{3a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2}$$

$$= \sqrt{\frac{9a^2}{4} + \frac{b^2}{4}}$$

and $$QS = \sqrt{\left(\frac{a}{2} - 2a\right)^2 + \left(\frac{b}{2} - 0\right)^2}$$

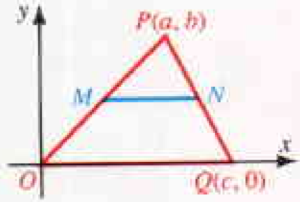
$$= \sqrt{\frac{9a^2}{4} + \frac{b^2}{4}}$$ Therefore, $\overline{OR} \cong \overline{QS}$.

It is possible to prove many theorems of geometry by using coordinate methods rather than the noncoordinate methods involving congruent triangles and angles formed by parallel lines. Coordinate proofs are sometimes, but not always, much easier than noncoordinate proofs. For example, compare the proof in Example 2 with the proof of Theorem 5-11 on page 178.

Example 2

Prove that the segment joining the midpoints of two sides of a triangle is parallel to the third side and is half as long as the third side.

Proof:



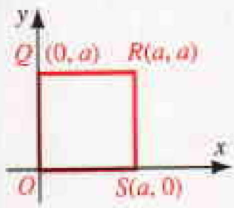
Let OPQ be any triangle. Choose convenient axes and coordinates as shown. By the midpoint formula, M has coordinates $\left(\frac{a}{2}, \frac{b}{2}\right)$ and N has coordinates $\left(\frac{a+c}{2}, \frac{b}{2}\right)$.

Slope of $\overline{MN} = 0$ and slope of $\overline{OQ} = 0$. (Why?)

Since $\overline{MN}$ and $\overline{OQ}$ have equal slopes, $\overline{MN} \parallel \overline{OQ}$.

Since $MN = \frac{a+c}{2} - \frac{a}{2} = \frac{c}{2}$ and $OQ = c - 0 = c$, $MN = \frac{1}{2}OQ$.

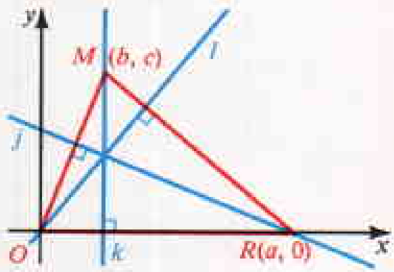
Classroom Exercises



In Exercises 1–4 use the diagram at the right.

1. What kind of figure is quad. $OQRS$? Why?
2. Show that $\overline{OR} \cong \overline{QS}$.
3. Show that $\overline{OR} \perp \overline{QS}$.
4. Show that $\overline{OR}$ bisects $\overline{QS}$.
5. The purpose of this exercise is to prove that the lines that contain the altitudes of a triangle intersect in a point (called the *orthocenter*).

   Given $\triangle ROM$, with lines j, k, and l containing the altitudes, we choose axes and coordinates as shown.



   a. The equation of line k is __?__.

   b. Since the slope of $\overline{MR}$ is $\frac{c}{b-a}$, the slope of line l is __?__.

   c. Show that an equation of line l is $y = \left(\frac{a-b}{c}\right)x$.

   d. Show that lines k and l intersect where $x = b$ and $y = \frac{ab - b^2}{c}$.

   e. Since the slope of $\overline{OM}$ = __?__, the slope of line j is __?__.

   f. Show that an equation of line j is $y = -\frac{b}{c}(x - a)$.

   g. Show that lines k and j intersect where $x = b$ and $y = \frac{ab - b^2}{c}$.

   h. From parts (d) and (g) we see that the three altitude lines intersect in a point. Name the coordinates of that point.

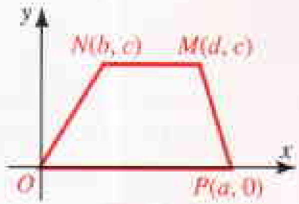
Written Exercises

Use coordinate geometry to prove each statement. First draw a figure and choose convenient axes and coordinates.

A

1. The diagonals of a rectangle are congruent. (Theorem 5-12)
2. The diagonals of a parallelogram bisect each other. (Theorem 5-3)
3. The diagonals of a rhombus are perpendicular. (Theorem 5-13) (*Hint*: Let the vertices be $(0, 0)$, $(a, 0)$, $(a + b, c)$, and (b, c). Show that $c^2 = a^2 - b^2$.)

Exercises 4–6 refer to trapezoid *MNOP* at the right.



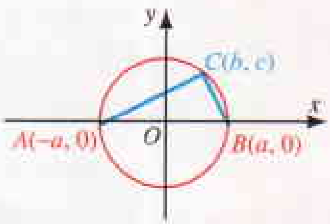
Exs. 4–6

4. Prove that the median of a trapezoid:
   a. is parallel to the bases.
   b. has a length equal to the average of the base lengths. (Theorem 5-19)
5. Prove that the segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases and has a length equal to half the difference of the lengths of the bases.
6. Assume that $a = b + d$.
   a. Show that the trapezoid is isosceles.
   b. Prove that its diagonals are congruent.

B

Ex. 7

7. Prove that the figure formed by joining, in order, the midpoints of the sides of quadrilateral *ROST* is a parallelogram.
8. Prove that the quadrilateral formed by joining, in order, the midpoints of the sides of an isosceles trapezoid is a rhombus.
9. Prove that an angle inscribed in a semicircle is a right angle. (*Hint*: The coordinates of *C* must satisfy the equation of the circle.)

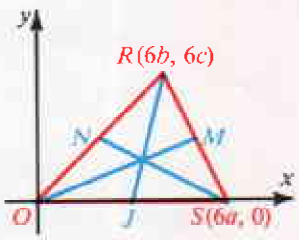


Ex. 9

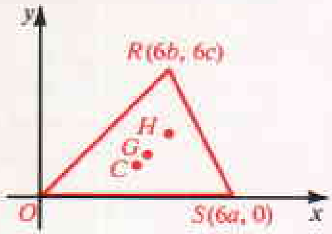
10. Prove that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals.

C

11. Use axes and coordinates as shown to prove: The medians of a triangle intersect in a point (called the *centroid*) that is two thirds of the distance from each vertex to the midpoint of the opposite side. (*Hint*: Find the coordinates of the midpoints, then the slopes of the medians, then the equations of the lines containing the medians.)



Exercises 12, 13, and 14 refer to the diagram at the right.



12. Prove that the perpendicular bisectors of the three sides of $\triangle ROS$ meet in a point C (called the *circumcenter*) whose coordinates are $\left(3a, \frac{3b^2 + 3c^2 - 3ab}{c}\right)$.

13. Prove that the lines containing the altitudes of $\triangle ROS$ intersect in a point $H\left(6b, \frac{6ab - 6b^2}{c}\right)$. (*Hint*: Use the procedure of Classroom Exercise 5.)

14. G, the intersection point of the medians of $\triangle ROS$, has coordinates $(2a + 2b, 2c)$. (See Exercise 11.)
Prove each statement.

a. Points C, G, and H are collinear. The line containing these points is called *Euler's Line*. (*Hint*: One way to prove this is to show that slope of $\overline{CG}$ = slope of $\overline{GH}$.)

b. $CG = \frac{1}{3}CH$

Self-Test 2

1. Find the slope and y-intercept of the line $2x - 5y = 20$.

2. Graph the line $2x + 3y = 6$.

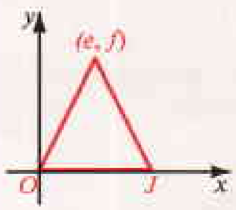
3. Write an equation of the line through $(1, 2)$ and $(5, 0)$.

4. Write an equation of the horizontal line through $(-2, 5)$.

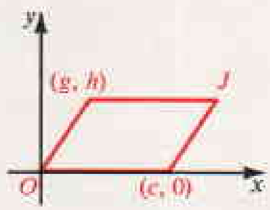
5. Find the intersection point of the lines $y = 3x - 4$ and $5x - 2y = 7$.

State the coordinates of point J without introducing any new letters.

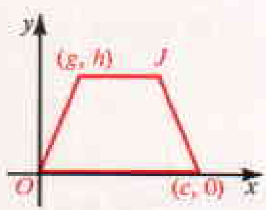
6. Isosceles triangle



7. Parallelogram



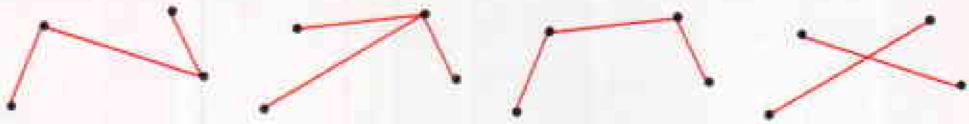
8. Isosceles trapezoid



9. The vertices of a quadrilateral are $G(4, -1)$, $O(0, 0)$, $L(2, 6)$, and $D(6, 5)$. Show that quadrilateral $GOLD$ is a parallelogram.

Application — *Steiner's Problem*

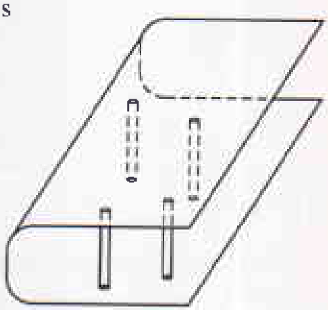
Four villages plan to build a system of roads of minimum length that will connect them all. Shown below are some plans for how to build the roads. Which plan shows the shortest road? Is there another way to connect the villages by an even shorter system of roads?



This problem was first investigated by the German mathematician Jacob Steiner (1796–1863), and carried his name, *Steiner's problem.*

Because a soap film automatically minimizes its surface area you can build a model that will help you solve Steiner's problem. You will need:

a sheet of clear plastic
8 split-pin paper fasteners
a drinking straw cut into four 3-cm long pieces



Bend the sheet of plastic without creasing it. Cut four small slits (to represent the location of the four villages) through both layers of the sheet. Insert the paper fasteners through all eight slits. Slip two fasteners through each of four straws, so that your model looks like the figure at the right. The halves of the plastic sheet should be parallel, and the straws perpendicular to them.

Dip the model in a soap solution and carefully lift it out. You should see a system of vertical soap films between the two sheets of plastic and joining the straws, revealing the solution to the problem. (Should any soap film adhere to the curved part of the plastic sheet, wet a drinking straw with the soap solution and push the straw through the soap films. You can suck air out through the straw to allow the films to form the minimum connection.)

Exercises

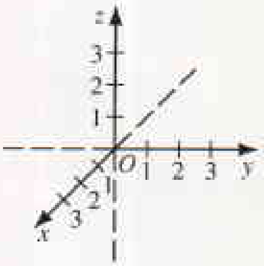
1. Gently place a protractor on top of the model and measure the angles where the soap films meet. What are the measures of these angles?

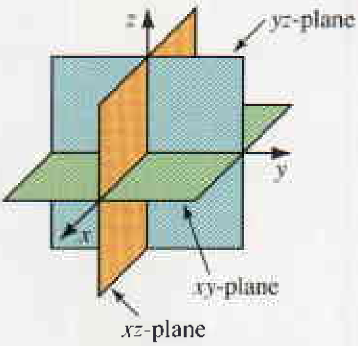
Make other models to find the shortest connection between the vertices of the following polygons. In each model, find the measures of the angles where the soap films meet.

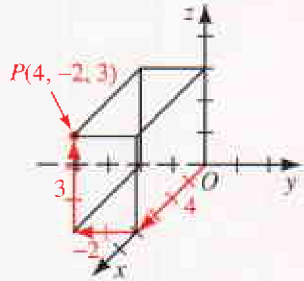
2. Triangle **3.** Square **4.** Pentagon

Extra — *Points in Space*

To locate points in three-dimensional space, three coordinate axes are needed. Think of the *y-axis* and *z-axis* as lying in the plane of the paper with the *x-axis* perpendicular to the plane of the paper. The axes intersect at the *origin*, or zero point, of each axis. The arrowhead on each axis indicates the positive direction.







The coordinate axes determine three *coordinate planes*, as shown in the middle diagram above. Each point in space has three coordinates: the *x-coordinate*, *y-coordinate*, and *z-coordinate*. For example, point P in the diagram at the right above, has coordinates (**4**, **−2**, **3**). The red arrows in the figure show that to *graph* P you start at O, move **4** units in the positive direction on the x-axis, **−2** units parallel to the y-axis (that is 2 units in the negative direction parallel to the y-axis), and **3** units in the positive direction parallel to the z-axis.

Exercises

On which axis or axes does each point lie?

1. (0, 7, 0) **2.** (0, 0, −9) **3.** (5, 0, 0) **4.** (0, 0, 0)

On which coordinate plane or planes does each point lie?

5. (1, −3, 0) **6.** (−7, 0, −1) **7.** (0, 8, 5) **8.** (0, 0, 0)

Graph each point on a coordinate system in space.

9. (−1, 4, 0) **10.** (2, 3, 1) **11.** (−2, −3, 4) **12.** (0, 1, −5)

Sketch the triangle in space whose vertices have the given coordinates.

13. (4, 0, 0), (0, 8, 0), (0, 0, 2) **14.** (1, 0, 0), (0, −5, 0), (0, 0, −5)

15. (−3, 0, 0), (0, −4, 0), (0, 0, 6) **16.** (0, 0, 0), (3, 0, 3), (0, −4, 5)

Chapter Summary

1. The distance between points (x_1, y_1) and (x_2, y_2) is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
The midpoint of the segment joining these points is the point
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

2. The circle with center (a, b) and radius r has the equation
$$(x - a)^2 + (y - b)^2 = r^2.$$

3. The slope m of a line through two points (x_1, y_1) and (x_2, y_2), $x_1 \neq x_2$, is defined as follows: $m = \frac{y_2 - y_1}{x_2 - x_1}$. The slope of a horizontal line is zero. Slope is not defined for vertical lines.

4. Two nonvertical lines with slopes m_1 and m_2 are:
   a. parallel if and only if $m_1 = m_2$.
   b. perpendicular if and only if $m_1 \cdot m_2 = -1$.

5. Any quantity that has both magnitude and direction is called a vector. A vector can be represented by an arrow or by an ordered pair. The magnitude of $\overrightarrow{AB}$ equals the length of $\overline{AB}$. Two vectors are perpendicular if the arrows representing them are perpendicular. Two vectors are parallel if the arrows representing them have the same or opposite directions. Two vectors are equal if they have the same magnitude and direction.

6. Two operations with vectors were discussed: multiplication of a vector by a real number, and addition of vectors.

7. The graph of any equation that can be written in the form $Ax + By = C$, with A and B not both zero, is a line. An equation of the line through point (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$. An equation of the line with slope m and y-intercept b is $y = mx + b$. The coordinates of the point of intersection of two lines can be found by solving their equations simultaneously.

8. To prove theorems using coordinate geometry, proceed as follows:
   a. Place x- and y-axes in a convenient position with respect to a figure.
   b. Use known properties to assign coordinates to points of the figure.
   c. Use the distance formula, the midpoint formula, and the slope properties of parallel and perpendicular lines to prove theorems.

Chapter Review

Exercises 1 and 2 refer to points $X(-2, -4)$, $Y(2, 4)$, and $Z(2, -6)$.

1. Graph X, Y, and Z on one set of axes, then find XY, YZ, and XZ. 13–1

2. Use the distance formula to show that $\triangle XYZ$ is a right triangle.

Find the center and radius of each circle.

3. $(x + 3)^2 + y^2 = 100$ **4.** $(x - 5)^2 + (y + 1)^2 = 49$

5. Write an equation of the circle that has center $(-6, -1)$ and radius 3.

6. Find the slope of the line through $(-5, -1)$ and $(15, -6)$. 13–2

7. A line with slope $\frac{2}{3}$ passes through $(9, -13)$ and $(0, \underline{\ ?\ })$.

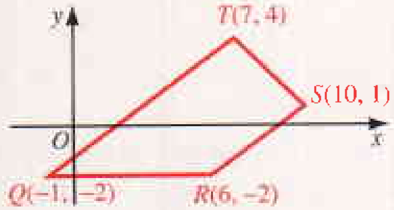
8. A line through $(0, -2)$ has slope 5. Find three other points on the line.

9. What is the slope of a line that is parallel to the x-axis?

10. Show that $QRST$ is a trapezoid. 13–3

11. Since the slope of $\overline{QT}$ is $\underline{\ ?\ }$, the slope of an altitude to $\overline{QT}$ is $\underline{\ ?\ }$.

12. If U is a point on $\overline{QT}$ such that $\overline{UR} \parallel \overline{ST}$, then U has coordinates $(\underline{\ ?\ }, \underline{\ ?\ })$.



13. Given points $P(3, -2)$ and $Q(7, 1)$, find **(a)** $\overrightarrow{PQ}$, **(b)** $|\overrightarrow{PQ}|$, and **(c)** $-2\overrightarrow{PQ}$. 13–4

14. Find the vector sum $(2, 6) + 3(1, -2)$ and illustrate with a diagram.

Find the coordinates of the midpoint of the segment that joins the given points.

15. $(7, -2)$ and $(1, -1)$ **16.** $(-4, 5)$ and $(2, -5)$ **17.** (a, b) and $(-a, b)$ 13–5

18. $M(0, 5)$ is the midpoint of $\overline{RS}$. If S has coordinates $(11, -1)$, then R is point $(\underline{\ ?\ }, \underline{\ ?\ })$.

19. Graph the line $y = 2x - 3$. **20.** Graph the line $x + 2y = 4$. 13–6

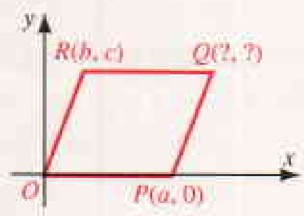
21. Find the point of intersection of the two lines in Exercises 19 and 20.

22. Find an equation of the line with slope 4 and y-intercept 7. 13–7

23. Find an equation of the line through $(-1, 2)$ and $(3, 10)$.

24. If $OPQR$ is a parallelogram, what are the coordinates of Q? 13–8

25. Let M be the midpoint of $\overline{RQ}$ and N be the midpoint of $\overline{OP}$. Use coordinate geometry to prove that $ONQM$ is a parallelogram. 13–9



Chapter Test

Given: Points $M(-2, 1)$ and $N(2, 4)$

1. Find **(a)** MN, **(b)** the slope of $\overline{MN}$, and **(c)** the midpoint of $\overline{MN}$.
2. Write an equation of $\overleftrightarrow{MN}$.
3. Write an equation of a circle with center M and radius MN.
4. If M is the midpoint of $\overline{NZ}$, what are the coordinates of Z?

In Exercises 5–8 write an equation of each line described.

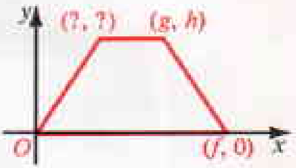
5. The line with slope $-\frac{3}{2}$ and y-intercept 4
6. The line with y-intercept 5 and x-intercept 3
7. The line through $(-2, 5)$ and parallel to $3x + y = 6$
8. The line with y-intercept 7 and perpendicular to $y = -2x + 3$
9. Given points $P(-2, 5)$ and $Q(4, 1)$, find **(a)** $\overrightarrow{PQ}$ and **(b)** $|\overrightarrow{PQ}|$.
10. The vectors $(3, 6)$ and $(-2, k)$ are parallel. Find the value of k.
11. The vectors $(3, -5)$ and $(c, 6)$ are perpendicular. Find the value of c.
12. Evaluate the vector sum $(5, -3) + 4(-2, 1)$.
13. Find the point of intersection of the lines $x + 2y = 8$ and $3x - y = 3$.

Draw the graph of each equation.

14. $2x - 3y = 6$
15. $y = 5$

16. Name 3 points on the line through $(2, 2)$ with slope $\frac{4}{3}$.

17. An isosceles trapezoid is shown. Give the missing coordinates without introducing any new letters.



Use points $J(-12, 0)$, $K(0, 6)$, and $L(-3, -3)$.

18. Show that $\triangle JKL$ is isosceles.
19. Use slopes to show that $\triangle JKL$ is a right triangle.

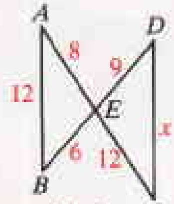
Use coordinate geometry to prove each statement.

20. The diagonals of a rectangle bisect each other.
21. The segments joining the midpoints of consecutive sides of a rectangle form a rhombus.

Cumulative Review: Chapters 1–13

A

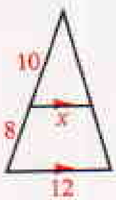
1. $\overrightarrow{BD}$ bisects $\angle ABC$, $m\angle ABC = 5x - 4$, and $m\angle CBD = \frac{3}{2}x + 21$. Is $\angle ABC$ acute, obtuse, or right?
2. Name five ways to prove that two lines are parallel.
3. If the diagonals of a quadrilateral are congruent and perpendicular, must the quadrilateral be a square? a rhombus? Draw a diagram to illustrate your answer.
4. Write "$x = 1$ only if $x \neq 0$" in if-then form. Then write the contrapositive and classify the contrapositive as true or false.
5. Refer to the diagram.
   a. Show that $\angle B \cong \angle D$.
   b. Find the value of x.
   c. Find the ratio of the areas of the triangles.

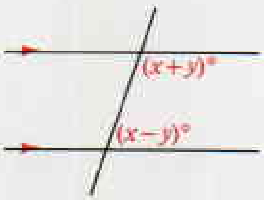


Ex. 5

6. Is a triangle with sides of lengths 12, 35, and 37 acute, right, or obtuse?
7. In $\triangle ABC$, $\overline{AB} \perp \overline{BC}$, $AB = 1$, and $AC = 3$. Find:
   a. $\cos A$ b. $\sin C$ c. $\tan A$ d. $\cos C$
8. Find the perimeter and area of a regular hexagon with apothem $\sqrt{3}$ cm.
9. Find the total area and volume of a cylinder with radius 10 and height 8.2.
10. Describe the locus of the centers of all circles tangent to each of two given parallel lines.

Find the value of x.

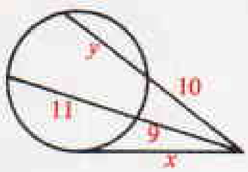
11.


12.


13.


B

14. If x is the length of a tangent segment in the diagram, find the values of x and y.



15. Prove: If the ray that bisects an angle of a triangle is perpendicular to the side that it intersects, then the triangle is an isosceles triangle.
16. Draw an obtuse triangle. Construct a circumscribed circle about the triangle.
17. Use coordinate geometry to prove that the median of a trapezoid is parallel to each base.