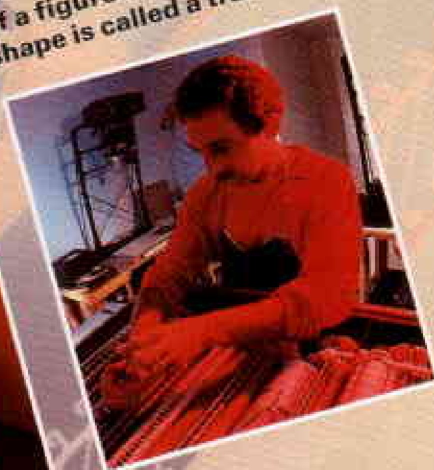
14 TRANSFORMATIONS

This striking color photograph shows a repeated design, identical balconies on one face of a building. A mathematical operation that changes the position of a figure without changing its shape is called a *transformation*.

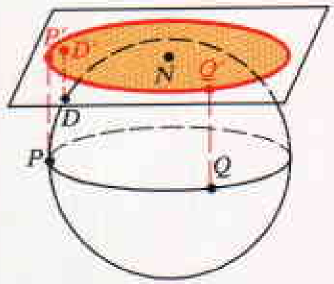


Some Basic Mappings

Objectives

1. Recognize and use the terms *image*, *preimage*, *mapping*, *one-to-one mapping*, *transformation*, *isometry*, and *congruence mapping*.
2. Locate images of figures by reflection, translation, glide reflection, rotation, and dilation.
3. Recognize the properties of the basic mappings.

14-1 *Mappings and Functions*



Have you ever wondered how maps of the round Earth can be made on flat paper? The diagram illustrates the idea behind a *polar map* of the northern hemisphere. A plane is placed tangent to a globe of the Earth at its North Pole N. Every point P of the globe is projected straight upward to exactly one point, called P', in the plane. P' is called the **image** of P, and P is called the **preimage** of P'. The diagram shows the images of two points P and Q on the globe's equator. It also shows D', the image of a point D not on the equator.

This correspondence between points of the globe's northern hemisphere and points in the plane is an example of a *mapping*. If we call this mapping M, then we could indicate that M maps P to P' by writing $M: P \rightarrow P'$. Notice that since the North Pole N is mapped to itself, we can write $M: N \rightarrow N$.

The word *mapping* is used in geometry as the word *function* is used in algebra. While a **mapping** is a correspondence between sets of points, a **function** is a correspondence between sets of numbers. Each number in the first set corresponds to exactly one number in the second set. For example, the squaring function f maps each real number x to its square x^2. We can write $f: x \rightarrow x^2$. Another way to indicate that the value of the function at x is x^2 is to write $f(x) = x^2$ (read "f of x equals x^2"). Similarly, for the mapping M, above, we can write $M(P) = P'$ to indicate that the image of P is P'. With all of these similarities, it should not surprise you that mathematicians often use the words *function* and *mapping* interchangeably.

A mapping (or a function) from set A to set B is called a **one-to-one mapping** (or a one-to-one function) if every member of B has exactly one preimage in A. The polar projection illustrated at the top of the page is a one-to-one mapping of the northern hemisphere of the globe onto a circular region in the tangent plane (the shaded area in the diagram). However, the squaring function $f: x \rightarrow x^2$ is *not* one-to-one because, for example, 9 has two preimages, 3 and -3.

Example 1 Function g maps every number to a number that is six more than its double.

a. Express this fact using function notation.
b. Find the image of 7.
c. Find the preimage of 8.

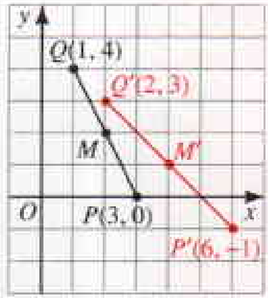
Solution

a. $g: x \to 2x + 6$, or $g(x) = 2x + 6$
b. $g: 7 \to 2 \cdot 7 + 6 = 20$. Thus the image of 7 is 20.
c. $g: x \to 2x + 6 = 8$. Therefore $x = 1$, so 1 is the preimage of 8.

Example 2 Mapping G maps each point (x, y) to the point $(2x, y - 1)$.

a. Express this fact using mapping notation.
b. Find P' and Q', the images of $P(3, 0)$ and $Q(1, 4)$.
c. Decide whether G maps M, the midpoint of $\overline{PQ}$, to M', the midpoint of $\overline{P'Q'}$.
d. Decide whether $PQ = P'Q'$.

Solution



a. $G: (x, y) \to (2x, y - 1)$

b. $G: (3, 0) \to (2 \cdot 3, 0 - 1) = (6, -1) = P'$
$G: (1, 4) \to (2 \cdot 1, 4 - 1) = (2, 3) = Q'$

c. $M = \left(\frac{3 + 1}{2}, \frac{0 + 4}{2}\right) = (2, 2)$

$M' = \left(\frac{6 + 2}{2}, \frac{-1 + 3}{2}\right) = (4, 1)$

$G: (2, 2) \to (2 \cdot 2, 2 - 1) = (4, 1)$
Thus G does map midpoint M to midpoint M'.

d. Use the distance formula to show that

$$PQ = \sqrt{(1 - 3)^2 + (4 - 0)^2} = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

$$P'Q' = \sqrt{(2 - 6)^2 + (3 - (-1))^2} = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

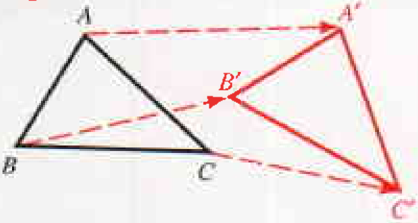
Thus $PQ \neq P'Q'$.

Although the diagram for Example 2 shows only points of $\overline{PQ}$ and their image points, you should understand that mapping G maps *every* point of the plane to an image point. Also, every point of the plane has a preimage point. A one-to-one mapping from the whole plane to the whole plane is called a **transformation.** Moreover, if a transformation maps every segment to a congruent segment, it is called an **isometry.** The transformation in Example 2 is *not* an isometry because $PQ \neq P'Q'$.

By definition, an isometry maps any segment to a congruent segment, so we can say that an isometry *preserves* distance. The next theorem states that an isometry also maps any triangle to a congruent triangle. For this reason, an isometry is sometimes called a **congruence mapping.**

Theorem 14-1

An isometry maps a triangle to a congruent triangle.



Given: Isometry $T: \triangle ABC \rightarrow \triangle A'B'C'$

Prove: $\triangle ABC \cong \triangle A'B'C'$

Proof:

Statements	Reasons
1. $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$	1. Definition of isometry
2. $\triangle ABC \cong \triangle A'B'C'$	2. SSS Postulate

Corollary 1

An isometry maps an angle to a congruent angle.

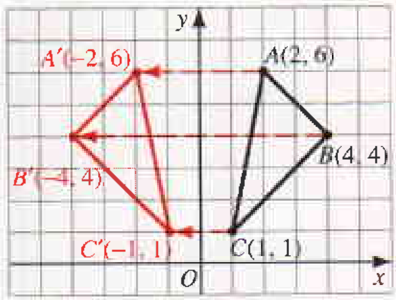
Corollary 2

An isometry maps a polygon to a polygon with the same area.

Example 3 Mapping R maps each point (x, y) to an image point $(-x, y)$.

a. Decide if $BA = B'A'$, $CB = C'B'$, and $CA = C'A'$.

b. Does R *appear* to be an isometry? Does part (a) *prove* that R is an isometry? Explain.



Solution **a.** Use the distance formula to show that

$$BA = \sqrt{(2-4)^2 + (6-4)^2} = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$B'A' = \sqrt{(-2-(-4))^2 + (6-4)^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$CB = \sqrt{(4-1)^2 + (4-1)^2} = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$C'B' = \sqrt{(-4-(-1))^2 + (4-1)^2} = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

$$CA = \sqrt{(2-1)^2 + (6-1)^2} = \sqrt{1^2 + 5^2} = \sqrt{26}$$

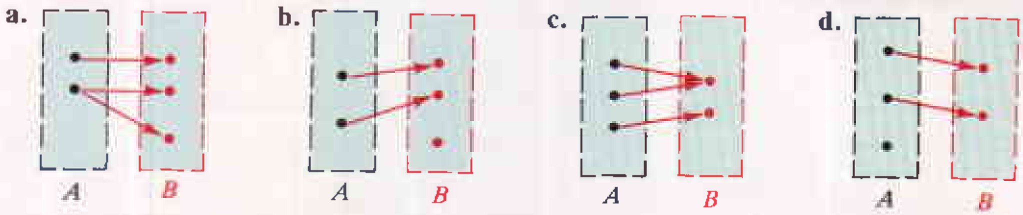
$$C'A' = \sqrt{(-2-(-1))^2 + (6-1)^2} = \sqrt{(-1)^2 + 5^2} = \sqrt{26}$$

We have $BA = B'A'$, $CB = C'B'$, and $CA = C'A'$.

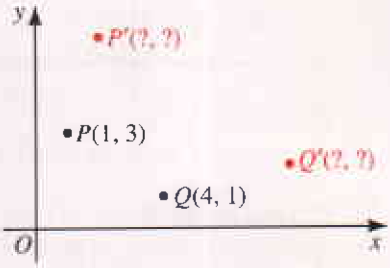
b. R *appears* to be an isometry because part (a) shows that three segments are mapped to congruent segments. Part (a) *does not prove* that R is an isometry because a proof must show that the image of *every* segment is a congruent segment.

Classroom Exercises

1. Explain why each of the correspondences pictured below is not a one-to-one mapping from set A to set B.



2. a. If $f: x \rightarrow |x|$, find the images of -3, 6, and -6.
   b. Is f a one-to-one function? Explain.



3. a. If mapping $M:(x, y) \rightarrow (2x, 2y)$, find the images of P and Q in the diagram.
   b. Is M a transformation?
   c. Does M appear to be an isometry?
   d. Decide whether M maps the midpoint of $\overline{PQ}$ to the midpoint of $\overline{P'Q'}$.
4. a. If $g(x) = 2x - 1$, find $g(8)$ and $g(-8)$.
   b. Find the image of 5.
   c. Find the preimage of 7.
5. Use the transformation $T:(x, y) \rightarrow (x + 1, y + 2)$ in this exercise.
   a. Plot the following points and their images on the chalkboard: $A(0, 0)$, $B(3, 4)$, $C(5, 1)$, and $D(-1, -3)$.
   b. Find AB, $A'B'$, CD, and $C'D'$.
   c. Does this transformation appear to be an isometry?
   d. What is the preimage of $(0, 0)$? of $(4, 5)$?

Exercises 6–8 refer to the globe shown on page 571.

6. What is the image of point N?
7. Is the distance between N and P on the globe the same as the corresponding distance on the polar map?
8. Does the polar map preserve or distort distances?
9. Explain how Corollary 1 follows from Theorem 14-1.
10. Explain how Corollary 2 follows from Theorem 14-1.

Written Exercises

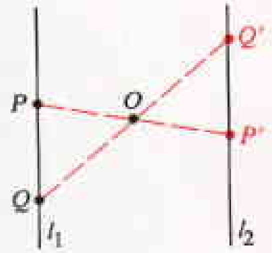
A

1. If function $f: x \rightarrow 5x - 7$, find the image of 8 and the preimage of 13.
2. If function $g: x \rightarrow 8 - 3x$, find the image of 5 and the preimage of 0.
3. If $f(x) = x^2 + 1$, find $f(3)$ and $f(-3)$. Is f a one-to-one function?
4. If $h(x) = 6x + 1$, find $h(\frac{1}{2})$. Is h a one-to-one function?

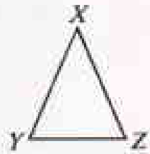
For each transformation given in Exercises 5–10:
a. Plot the three points $A(0, 4)$, $B(4, 6)$, and $C(2, 0)$ and their images A', B', and C' under the transformation.
b. State whether the transformation appears to be an isometry.
c. Find the preimage of (12, 6).

5. $T:(x, y) \rightarrow (x + 4, y - 2)$
6. $S:(x, y) \rightarrow (2x + 4, 2y - 2)$
7. $D:(x, y) \rightarrow (3x, 3y)$
8. $H:(x, y) \rightarrow (-x, -y)$
9. $M:(x, y) \rightarrow (12 - x, y)$
10. $G:(x, y) \rightarrow (-\frac{1}{2}x, -\frac{1}{2}y)$

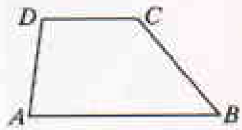
11. O is a point equidistant from parallel lines l_1 and l_2. A mapping M maps each point P of l_1 to the point P' where $\overrightarrow{PO}$ intersects l_2.
   a. Is the mapping a one-to-one mapping of l_1 onto l_2?
   b. Does this mapping preserve or distort distance?
   c. If l_1 and l_2 were not parallel, would the mapping preserve distance? Illustrate your answer with a sketch.



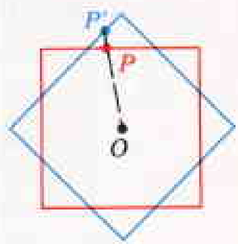
12. $\triangle XYZ$ is isosceles with $\overline{XY} \cong \overline{XZ}$. Describe a way of mapping each point of $\overline{XY}$ to a point of $\overline{XZ}$ so that the mapping is an isometry.



B 13. $ABCD$ is a trapezoid. Describe a way of mapping each point of $\overline{DC}$ to a point of $\overline{AB}$ so that the mapping is one-to-one. Is your mapping an isometry?

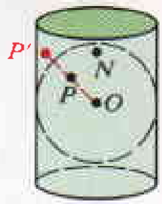


14. The red and blue squares are congruent and have the same center O. A mapping maps each point P of the red square to the point P' where $\overrightarrow{OP}$ intersects the blue square.
   a. Is this mapping one-to-one?
   b. Copy the diagram and locate a point X that is its own image.
   c. Locate two points R and S on the red square and their images R' and S' on the blue square that have the property that $RS \neq R'S'$.
   d. Does this mapping preserve distance?
   e. Describe a mapping from the red square onto the blue square that *does* preserve distance.



15. The transformation $T:(x, y) \rightarrow (x + y, y)$ preserves areas of figures even though it does not preserve distances. Illustrate this by drawing a square with vertices $A(2, 3)$, $B(4, 3)$, $C(4, 5)$, and $D(2, 5)$ and its image $A'B'C'D'$. Find the area and perimeter of each figure.

A piece of paper is wrapped around a globe of the Earth to form a cylinder as shown. O is the center of the Earth and a point P of the globe is projected along $\overrightarrow{OP}$ to a point P' of the cylinder.



16. Describe the image of the globe's equator.

17. Is the image of the Arctic Circle congruent to the image of the equator?

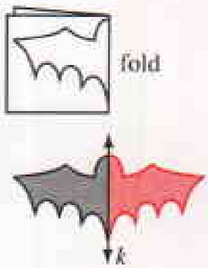
18. Are distances near the equator distorted more than or less than distances near the Arctic Circle?

19. Does the North Pole (point N) have an image?

20. Consider the mapping $S: (x, y) \rightarrow (x, 0)$.
   a. Plot the points $P(4, 5)$, $Q(-3, 2)$, and $R(-3, -1)$ and their images.
   b. Does S appear to be an isometry? Explain.
   c. Is S a transformation? Explain.

21. Mapping M maps points A and B to the same image point. Explain why the mapping M does not preserve distance.

22. Fold a piece of paper. Cut a design connecting the top and bottom point of the fold, as shown. Unfold the shape. Consider a mapping M of the points in the gray region to the corresponding points in the red region.
   a. Does M appear to be an isometry?
   b. If a point P is on line k, what is the image of P?
   c. If a point Q is not on line k, and $M(Q) = Q'$, what is the relationship between line k and $\overline{QQ'}$?



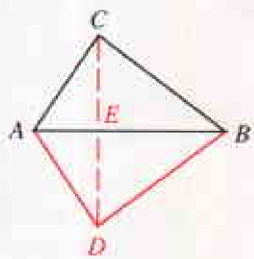
C **23.** **a.** Plot the points $A(6, 1)$, $B(3, 4)$, and $C(1, -3)$ and their images A', B', and C' under the transformation $R: (x, y) \rightarrow (-x, y)$.
   b. Prove that R is an isometry. (*Hint*: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Find P' and Q', and use the distance formula to show that $PQ = P'Q'$.)

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

As you will learn in the next lesson, a *reflection* is a mapping in the plane across a mirror line, just as your reflection in a mirror is a mapping in space across a mirror plane.

Draw any $\triangle ABC$. Reflect C in $\overleftrightarrow{AB}$ to locate point D. Draw $\overline{AD}$ and $\overline{BD}$. What do you notice about $\triangle ABC$ and $\triangle ABD$?

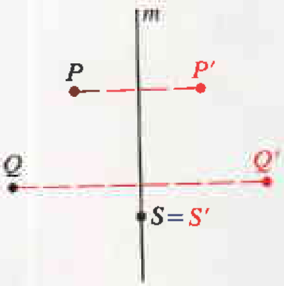


Draw $\overline{CD}$. Label the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ as E. Compare CE and DE. What do you notice? Measure the angles with vertex E. What do you notice?

Repeat the construction with other types of triangles.

14-2 *Reflections*

When you stand before a mirror, your image appears to be as far behind the mirror as you are in front of it. The diagram shows a transformation in which a line acts like a mirror. Points P and Q are reflected in line m to their images P' and Q'. This transformation is called a *reflection*. Line m is called the *line of reflection* or the mirror line.



A **reflection** in line m maps every point P to a point P' such that:

(1) If P is not on the line m, then m is the perpendicular bisector of $\overline{PP'}$.

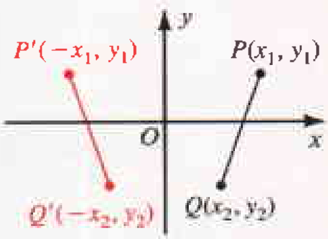
(2) If P is on line m, then $P' = P$.

To abbreviate *reflection in line m*, we write R_m. To abbreviate the statement *R_m maps P to P'*, we write $R_m: P \rightarrow P'$ or $R_m(P) = P'$. This may also be read as *P is reflected in line m to P'*.

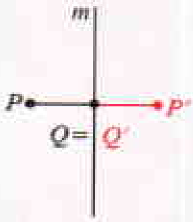
Theorem 14-2

A reflection in a line is an isometry.

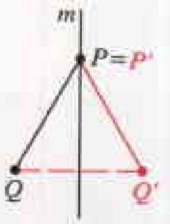
To prove Theorem 14-2 by using coordinates, we assign coordinates in the plane so that the line of reflection becomes the y-axis. Then in coordinate terms the reflection is $R: (x_1, y_1) \rightarrow (-x_1, y_1)$. In Exercise 23 on page 576 the distance formula was used to prove that $PQ = P'Q'$. Although the diagram shows P and Q on the same side of the y-axis, you should realize that the coordinates x_1, y_1, x_2, and y_2 can be positive, negative, or zero, thereby covering all cases.



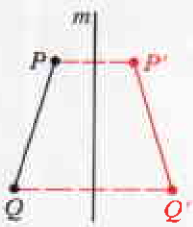
Theorem 14-2 can also be proved without the use of coordinates. If coordinates are not used, we must show that $PQ = P'Q'$ for all choices of P and Q. Four of the possible cases are shown below. In Written Exercises 18–20 you will prove Theorem 14-2 for Cases 2–4, using the fact that the line of reflection, m, is the perpendicular bisector of $\overline{PP'}$ and $\overline{QQ'}$.



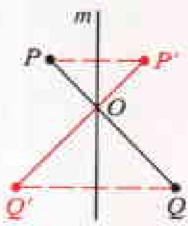
Case 1



Case 2

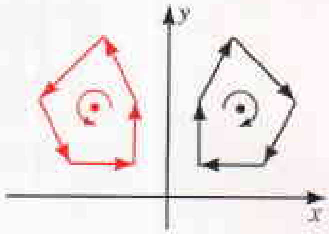


Case 3



Case 4

Since a reflection is an isometry, it preserves distance, angle measure, and the area of a polygon. Another way to say this is that distance, angle measure, and area are *invariant* under a reflection. On the other hand, the orientation of a figure is *not* invariant under a reflection because a reflection changes a clockwise orientation to a counterclockwise one, as shown at the right.



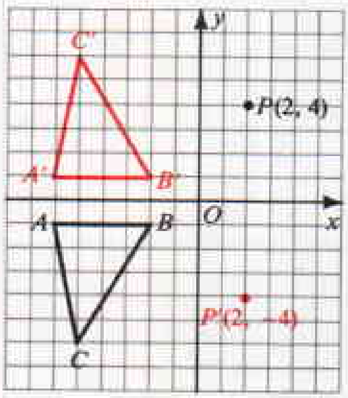
Example Find the image of point $P(2, 4)$ and $\triangle ABC$ under each reflection.

a. The line of reflection is the x-axis.

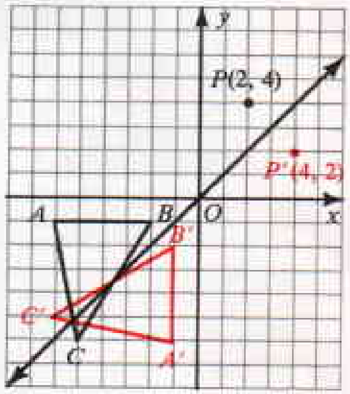
b. The line of reflection is the line $y = x$.

Solution The images are shown in red.

a.



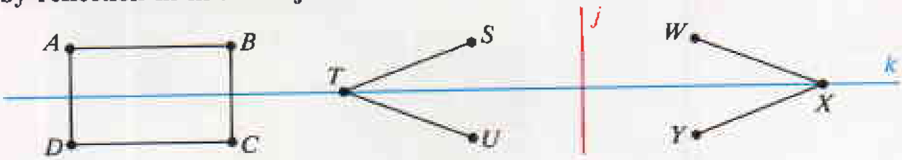
b.



Notice that under reflection in the line $y = x$, the point (x, y) is mapped to the point (y, x).

Classroom Exercises

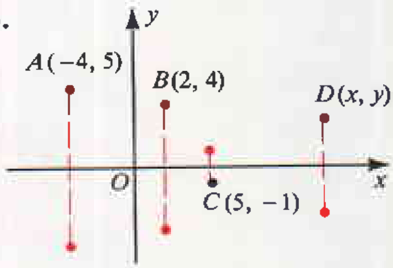
Complete the following. Assume points *D*, *C*, *U*, *W*, *X*, and *Y* are obtained by reflection in line *k* or *j*.



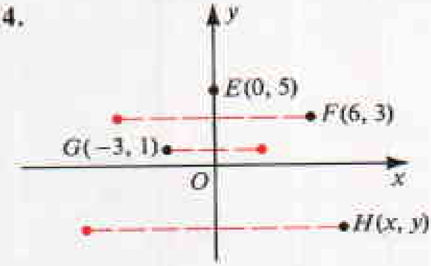
1. R_k stands for __?__.
2. $R_k: A \rightarrow$ __?__
3. $R_k(B) =$ __?__
4. $R_k: \overline{AB} \rightarrow$ __?__
5. $R_k(C) =$ __?__
6. $R_k: T \rightarrow$ __?__
7. $R_k: \overline{BC} \rightarrow$ __?__
8. $R_k: \angle STU \rightarrow$ __?__
9. $R_j(S) =$ __?__
10. $R_j: \overline{ST} \rightarrow$ __?__
11. $R_j($__?__$) = \overline{XY}$
12. R_j: line $k \rightarrow$ __?__

Points *A–D* are reflected in the *x*-axis. Points *E–H* are reflected in the *y*-axis. State the coordinates of the images.

13.



14.



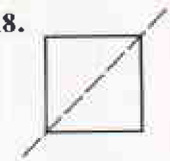
Sketch each figure on the chalkboard. With a different color, sketch its image, using the dashed line as the line of reflection.

15.

16.

17.

18.

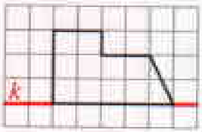


19. Under a reflection, is an angle always mapped to a congruent angle? Is a polygon always mapped to a polygon with the same area? Explain.
20. Explain in your own words the meaning of each phrase.
    - **a.** An isometry preserves distance.
    - **b.** Area is invariant under a reflection.
    - **c.** Orientation is not invariant under a reflection.

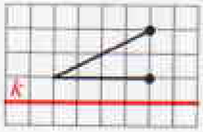
Written Exercises

Copy each figure on graph paper. Then draw the image by reflection in line *k*.

A 1.



2.



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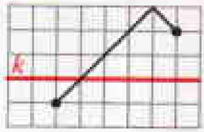

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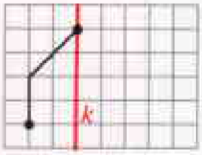

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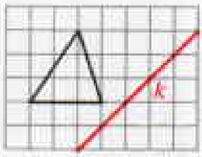

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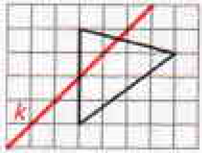
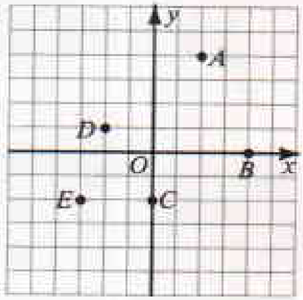


Write the coordinates of the image of each point by reflection in (a) the *x*-axis, (b) the *y*-axis, and (c) the line $y = x$. (*Hint:* Refer to the Example on page 578.)

7. *A* 8. *B* 9. *C*

10. *D* 11. *E* 12. *O*



Exs. 7–12

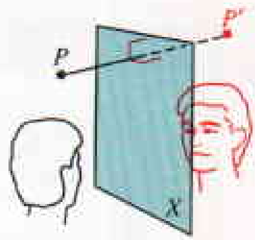
13. When the word MOM is reflected in a vertical line, the image is still MOM. Can you think of other words that are unchanged when reflected in a vertical line?

14. When the word HIDE is reflected in a horizontal line, the image is still HIDE. Can you think of other words that are unchanged when reflected in a horizontal line?

B 15. Draw a triangle and a line *m* such that R_m maps the triangle to itself. What kind of triangle did you use?

16. Draw a pentagon and a line *n* such that R_n maps the pentagon to itself.

17. The sketch illustrates a *reflection in plane X*. Write a definition of this reflection similar to the definition of a reflection in line *m* on page 577.



Ex. 17

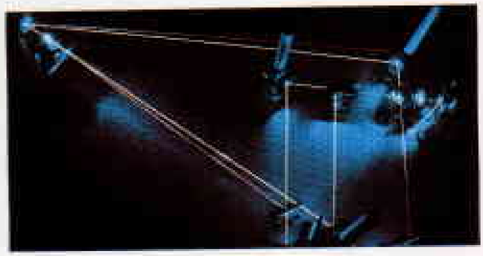
In Exercises 18–20, refer to the diagrams on page 578. Given the reflection $R_m : \overline{PQ} \rightarrow \overline{P'Q'}$, write the key steps of a proof that $PQ = P'Q'$ for each case.

18. Case 2 19. Case 3 20. Case 4

21. Draw a line *t* and a point *A* not on *t*. Then use a straightedge and compass to construct the image of *A* under R_t.

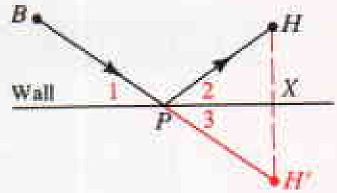
22. Draw any two points *B* and *B'*. Then use a straightedge and compass to construct the line of reflection *j* so that $R_j(B) = B'$.

23. If a transformation maps two parallel lines to two image lines that are also parallel, we say that parallelism is invariant under the transformation. Is parallelism invariant under a reflection?



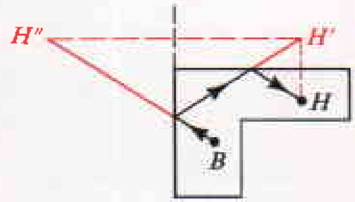
The photograph shows a reflected beam of laser light. Exercises 24–28 deal with the similar reflected path of a golf ball bouncing off the walls of a miniature golf layout. These exercises show how the geometry of reflections can be used to solve the problem of aiming a reflected path at a particular target.

24. A rolling ball that does not have much spin will bounce off a wall so that the two angles that the path forms with the wall are congruent. Thus, to roll the ball from B off the wall shown and into hole H, you need to aim the ball so that $\angle 1 \cong \angle 2$.

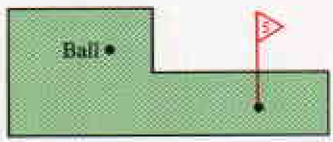


   a. Let H' be the image of H by reflection in the wall. $\overline{BH'}$ intersects the wall at P. Why is $\angle 1 \cong \angle 3$? Why is $\angle 3 \cong \angle 2$? Why is $\angle 1 \cong \angle 2$? You can conclude that if you aim for H', the ball will roll to H.
   b. Show that the distance traveled by the ball equals the distance BH'.

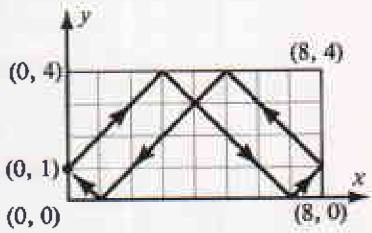
25. In the two-wall shot illustrated at the right, a reflection in one wall maps H to H', and a reflection in a second wall (extended) maps H' to H''. To roll the ball from B to H, you aim for H''. Show that the total distance traveled by the ball equals the distance BH''.



26. Show how to score a hole in one on the fifth hole of the golf course shown by rolling the ball off one wall.
27. Repeat Exercise 26 but roll the ball off two walls.
28. Repeat Exercise 26 but roll the ball off three walls.



29. A ball rolls at a 45° angle away from one side of a billiard table that has a coordinate grid on it. If the ball starts at the point (0, 1) it will eventually return to its starting point. Would this happen if the ball started from other points on the y-axis between (0, 0) and (0, 4)?



30. The line with equation $y = 2x + 3$ is reflected in the y-axis. Find an equation of the image line.

31. The line with equation $y = x + 5$ is reflected in the x-axis. Find an equation of the image line.

In each exercise $R_k : A \rightarrow A'$. Find an equation of line k.

	32.	33.	34.	35.	36.	37.
A	(5, 0)	(1, 4)	(4, 0)	(5, 1)	(0, 2)	(−1, 2)
A'	(9, 0)	(3, 4)	(4, 6)	(1, 5)	(4, 6)	(4, 5)

C **38.** Draw the x- and y-axes and the line l with equation $y = -x$. Plot several points and their images under R_l. What is the image of (a, b)?

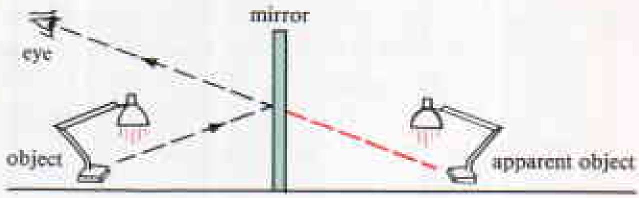
39. Draw the x- and y-axes and the vertical line j with equation $x = 5$. Find the images under R_j of the following points.

a. (4, 3) **b.** (0, −2) **c.** (−3, 1) **d.** (x, y)

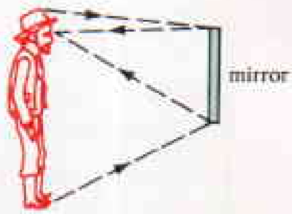
40. Repeat Exercise 39 letting j be the horizontal line with equation $y = 6$.

Application *Mirrors*

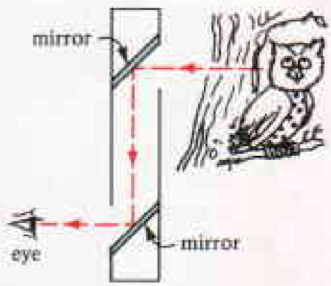
If a ray of light strikes a mirror at an angle of 40°, it will be reflected off the mirror at an angle of 40° also. The angle between the mirror and the reflected ray is always congruent to the angle between the mirror and the initial light ray. In the diagram at the left below, $\angle 2 \cong \angle 1$.



We see objects in a mirror when the reflected light ray reaches the eye. The object appears to lie behind the mirror as shown in the diagram at the right above.



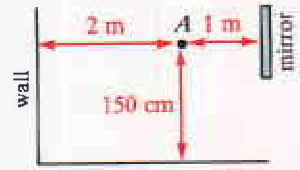
You don't need a full-length mirror to see all of yourself. A mirror that is only half as tall as you are will do if the mirror is in a position as shown. You see the top of your head at the top of the mirror and your feet at the bottom of the mirror. If the mirror is too high or too low, you will not see your entire body.



A periscope uses mirrors to enable a viewer to see above the line of sight. The diagram at the right is a simple illustration of the principle used in a periscope. It has two mirrors, parallel to each other, at the top and at the bottom. The mirrors are placed at an angle of 45° with the horizontal. Horizontal light rays from an object entering at the top are reflected down to the mirror at the bottom. They are then reflected to the eye of the viewer.

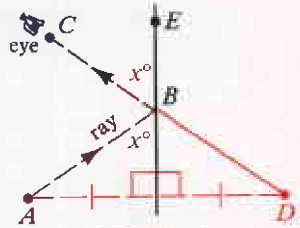
Exercises

1. What are the measures of the angles that the initial light ray and the reflected light rays make with the mirrors in the diagram of the periscope on the previous page?
2. If you can see the eyes of someone when you look into a mirror, can the other person see your eyes in that same mirror?
3. A person with eyes at A, 150 cm above the floor, faces a mirror 1 m away. The mirror extends 30 cm above eye level. How high can the person see on a wall 2 m behind point A?



Ex. 3

4. Prove that you can see all of yourself in a mirror that is only half as tall as you are. (*Hint*: Study the diagram on page 582.)
5. Prove that the point D which is as far behind the mirror as the object A is in front of the mirror lies on $\overleftrightarrow{BC}$. (*Hint*: Show that $\angle CBE$ and $\angle EBD$ are supplementary.)
6. Show that the light ray follows the shortest possible path from A to C via the mirror by proving that for any point E on the mirror (other than B) $AE + EC > AB + BC$. (*Hint*: See the Application: Finding the Shortest Path, on page 224.)

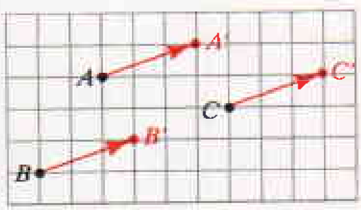


Exs. 5, 6

14-3 *Translations and Glide Reflections*

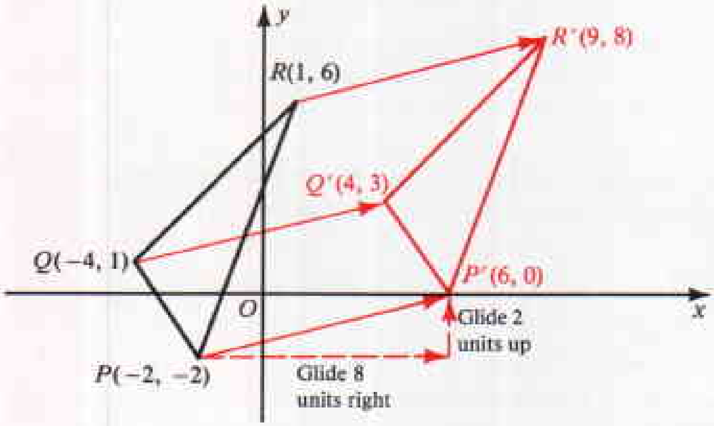
The photograph at the right suggests the transformation called a *translation*, or *glide*. The skate blades of the figure-skating pair move in identical ways when the pair is skating together. A transformation that glides all points of the plane the same distance in the same direction is called a **translation.**

If a transformation maps A to A', B to B', and C to C', points A, B, and C glide along parallel or collinear segments and $AA' = BB' = CC'$. Any of the vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, or $\overrightarrow{CC'}$ could describe this translation.



Each vector has the same magnitude of $\sqrt{1^2 + 3^2}$, or $\sqrt{10}$, and each vector has the same direction as indicated by its slope of $\frac{1}{3}$. Note that we don't need to know the coordinates of points A, B, or C to describe the translation. All that is important is the change in the x-coordinate and y-coordinate of each point.

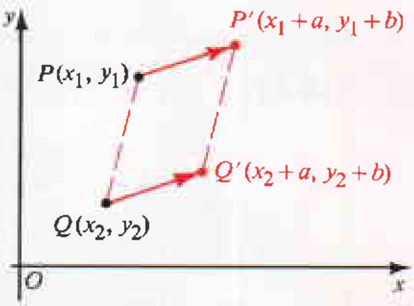
Consider a translation in which every point glides 8 units right and 2 units up. We could use the vector (8, 2) to indicate such a translation, or we could use the coordinate expression $T:(x, y) \rightarrow (x + 8, y + 2)$. The following diagram shows how $\triangle PQR$ is mapped by T to $\triangle P'Q'R'$. You can use the distance formula to check that each segment is mapped to a congruent segment so that T is an isometry.



The illustration just presented should help you to understand why we use the following definition of a translation when working in the coordinate plane. A **translation,** or glide, in a plane is a transformation T which maps any point (x, y) to the point $(x + a, y + b)$ where a and b are constants. This definition makes it possible to give a simple proof of the following theorem.

Theorem 14-3

A translation is an isometry.



Plan for Proof: Label two points P and Q and their images P' and Q' as shown in the diagram. To show that T is an isometry, we need to show that $PQ = P'Q'$. Use the distance formula to show that:

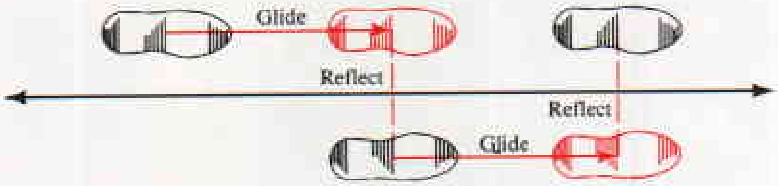
$$PQ = P'Q' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Of course, since a translation is an isometry, we know by the corollaries of Theorem 14-1, page 573, that a translation preserves angle measure and area.

A glide and a reflection can be carried out one after the other to produce a transformation known as a *glide reflection*. A **glide reflection** is a transformation in which every point P is mapped to a point P'' by these steps:

1. A glide maps P to P'.
2. A reflection in a line parallel to the glide line maps P' to P''.

A glide reflection combines two isometries to produce a new transformation, which is itself an isometry. The succession of footprints shown illustrates a glide reflection. Note that the reflection line is parallel to the direction of the glide.



As long as the glide is parallel to the line of reflection, it doesn't matter whether you glide first and then reflect, or reflect first and then glide. For other combinations of mappings, the order in which you perform the mappings will affect the result. We will look further at such combinations of mappings in Section 14-6.

Classroom Exercises

1. Complete each statement for the translation $T:(x, y) \to (x + 3, y - 1)$.
   a. T glides points __?__ units right and 1 unit __?__.
   b. The image of (4, 6) is (__?__, __?__).
   c. The preimage of (2, 3) is (__?__, __?__).

Describe each translation in words, as in Exercise 1(a), and give the image of (4, 6) and the preimage of (2, 3).

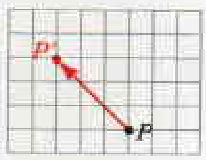
2. $T:(x, y) \to (x - 5, y + 4)$

3. $T:(x, y) \to (x + 1, y)$

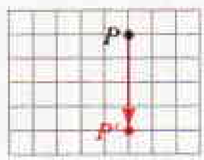
Each diagram shows a point P on the coordinate plane and its image P' under a translation T. Complete the statement $T:(x, y) \to$ (__?__, __?__).

4.

5.



6.



7. For a given translation, the image of the origin is (5, 7). What is the preimage of the origin?

8. A glide reflection has the glide translation $T:(x, y) \to (x + 2, y + 2)$. The line of reflection is line m with equation $y = x$.
   a. Find the image, point S', of $S(-1, 3)$ under T.
   b. Find the image, point S'', of S' under R_m. (*Hint*: Recall from the example on page 578 that $R_m:(x, y) \to (y, x)$.)
   c. Under the glide reflection, (x, y) is first mapped to (__?__, __?__) and then to (__?__, __?__).

Written Exercises

In Exercises 1 and 2 a translation T is described. For each:
a. Graph $\triangle ABC$ and its image $\triangle A'B'C'$. Is $\triangle ABC \cong \triangle A'B'C'$?
b. In color, draw arrows from A to A', B to B', and C to C'.
c. Are your arrows the same length? Are they parallel?

A

1. $T:(x, y) \to (x - 2, y + 6)$
   $A(-2, 0)$, $B(0, 4)$, $C(3, -1)$
2. $T:(x, y) \to (x - 3, y - 6)$
   $A(3, 6)$, $B(-3, 6)$, $C(-1, -2)$
3. If $T:(0, 0) \to (5, 1)$, then $T:(3, 3) \to (\underline{\ ?\ }, \underline{\ ?\ })$.
4. If $T:(1, 1) \to (3, 0)$, then $T:(0, 0) \to (\underline{\ ?\ }, \underline{\ ?\ })$.
5. If $T:(-2, 3) \to (2, 6)$, then $T:(\underline{\ ?\ }, \underline{\ ?\ }) \to (0, 0)$.
6. The image of $P(-1, 5)$ under a translation is $P'(5, 7)$. What is the preimage of P?

In each exercise a glide reflection is described. Graph $\triangle ABC$ and its image under the glide, $\triangle A'B'C'$. Also graph $\triangle A''B''C''$, the image of $\triangle A'B'C'$ under the reflection.

7. Glide: All points move up 4 units.
   Reflection: All points are reflected in the y-axis.
   $A(1, 0)$, $B(4, 2)$, $C(5, 6)$
8. Glide: All points move left 7 units.
   Reflection: All points are reflected in the x-axis.
   $A(4, 2)$, $B(7, 0)$, $C(9, -3)$

B

9. Where does the glide reflection in Exercise 7 map (x, y)?
10. Where does the glide reflection in Exercise 8 map (x, y)?
11. Which of the following properties are invariant under a translation?
    a. distance **b.** angle measure **c.** area **d.** orientation
12. Which of the properties listed in Exercise 11 are invariant under a glide reflection?

In Exercises 13 and 14 translations R and S are described. R maps point P to P', and S maps P' to P''. Find T, the translation that maps P to P''.

13. $R:(x, y) \to (x + 1, y + 2)$
    $S:(x, y) \to (x - 5, y + 7)$
    $T:(x, y) \to (\underline{\ ?\ }, \underline{\ ?\ })$
14. $R:(x, y) \to (x - 5, y - 3)$
    $S:(x, y) \to (x + 4, y - 6)$
    $T:(x, y) \to (\underline{\ ?\ }, \underline{\ ?\ })$
15. If a translation T maps P to P', then T can be described by the vector $\overrightarrow{PP'}$. Suppose a translation T is described by the vector $(3, -4)$ because it glides all points 3 units right and 4 units down.
    a. Graph points $A(-1, 2)$, $B(0, 6)$, A', and B', where $T(A) = A'$ and $T(B) = B'$.
    b. What kind of figure is $AA'B'B$? What is its perimeter?

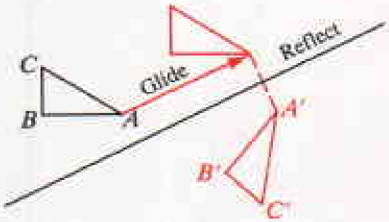
16. a. Graph $\triangle POQ$ with vertices $P(0, 3)$, $O(0, 0)$, and $Q(6, 0)$.

b. $T_1:(x, y) \rightarrow (x + 2, y - 4)$ and $T_2:(x, y) \rightarrow (x + 5, y + 6)$. If $T_1:\triangle POQ \rightarrow \triangle P'O'Q'$ and $T_2:\triangle P'O'Q' \rightarrow \triangle P''O''Q''$, graph $\triangle P'O'Q'$ and $\triangle P''O''Q''$.

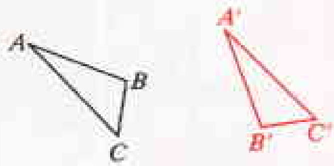
c. Find T_3, a translation that maps $\triangle POQ$ directly to $\triangle P''O''Q''$.

d. Because T_1 glides all points 2 units right and 4 units down, the translation can be described by the vector $\overrightarrow{T_1} = (2, -4)$. Describe T_2 and T_3 by vectors. How are these three vectors related?

17. A glide reflection maps $\triangle ABC$ to $\triangle A'B'C'$. Copy the diagram and locate the midpoints of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. What seems to be true about these midpoints? Try to prove your conjecture.

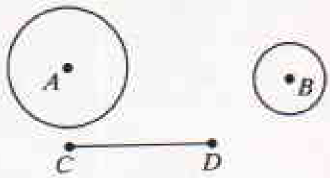


18. Copy the figure and use the result of Exercise 17 to construct the reflecting line of the glide reflection that maps $\triangle ABC$ to $\triangle A'B'C'$. Also construct the glide image of $\triangle ABC$.



19. Explain why a glide reflection is an isometry.

20. Given $\odot A$ and $\odot B$ and $\overline{CD}$, construct a segment $\overline{XY}$ parallel to and congruent to $\overline{CD}$ and having X on $\odot A$ and Y on $\odot B$. (*Hint*: Translate $\odot A$ along a path parallel to and congruent to $\overline{CD}$.)



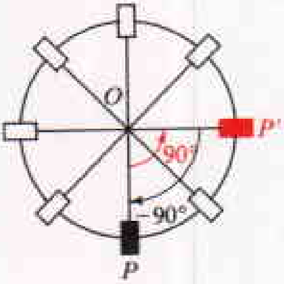
C 21. Describe how you would construct points X and Y, one on each of the lines shown, so that $\overline{XY}$ is parallel to and congruent to $\overline{EF}$.

22. Show by example that if a glide is not parallel to a line of reflection, then the image of a point when the glide is followed by the reflection will be different from the image of the same point when the reflection is followed by the glide.

23. Prove Theorem 14-3 (page 584).

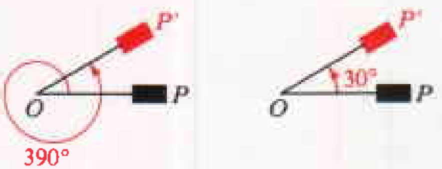
14-4 *Rotations*

A *rotation* is a transformation suggested by rotating a paddle wheel. When the wheel moves, each paddle rotates to a new position. When the wheel stops, the new position of a paddle (P') can be referred to mathematically as the image of the initial position of the paddle (P).

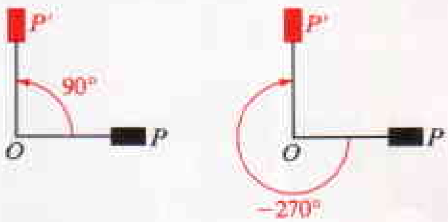


For the counterclockwise rotation shown about point O through 90°, we write $\mathcal{R}_{O,\,90}$. A counterclockwise rotation is considered positive, and a clockwise rotation is considered negative. If the red paddle is rotated about O clockwise until it moves into the position of the black paddle, the rotation is denoted by $\mathcal{R}_{O,\,-90}$. (Note that to avoid confusion with the R used for reflections we use a script $\mathcal{R}$ for rotations.)

A full revolution, or 360° rotation about point O, rotates any point P around to itself so that $P' = P$. The diagram at the left below shows a rotation of 390° about O. Since 390° is 30° more than one full revolution, the image of any point P under a 390° rotation is the same as its image under a 30° rotation, and the two rotations are said to be equal. Similarly, the diagram at the right below shows that a 90° counterclockwise rotation is equal to a 270° clockwise rotation because both have the same effect on any point P.



$\mathcal{R}_{O,\,390} = \mathcal{R}_{O,\,30}$
Notice: $390 - 360 = 30$



$\mathcal{R}_{O,\,90} = \mathcal{R}_{O,\,-270}$
Notice: $90 - 360 = -270$

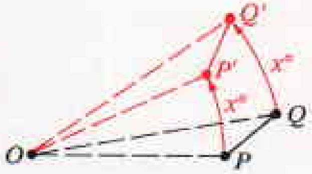
In the following definition of a rotation, the angle measure x can be positive or negative and can be more than 180 in absolute value.

A **rotation** about point O through $x°$ is a transformation such that:

(1) If a point P is different from O, then $OP' = OP$ and $m\angle POP' = x$.
(2) If point P is the point O, then $P' = P$.

Theorem 14-4

A rotation is an isometry.

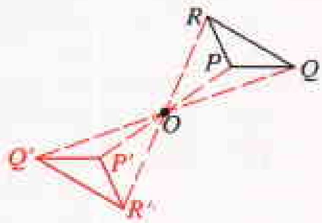


Given: $\mathcal{R}_{O,x}$ maps P to P' and Q to Q'.
Prove: $\overline{PQ} \cong \overline{P'Q'}$

Key steps of proof:

1. $OP = OP'$, $OQ = OQ'$ (Definition of rotation)
2. $m\angle POP' = m\angle QOQ' = x$ (Definition of rotation)
3. $m\angle POQ = m\angle P'OQ'$ (Subtraction Property of $=$: subtract $m\angle QOP'$.)
4. $\triangle POQ \cong \triangle P'OQ'$ (SAS Postulate)
5. $\overline{PQ} \cong \overline{P'Q'}$ (Corr. parts of $\cong$ $\triangle$s are $\cong$.)

A rotation about point O through 180° is called a **half-turn** about O and is usually denoted by H_O. The diagram shows $\triangle PQR$ and its image $\triangle P'Q'R'$ by H_O. Notice that O is the midpoint of $\overline{PP'}$, $\overline{QQ'}$, and $\overline{RR'}$.



Using coordinates, a half-turn H_O about the origin can be written

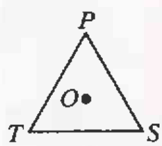
$$H_O:(x, y) \rightarrow (-x, -y).$$

Classroom Exercises

State another name for each rotation.

1. $\mathcal{R}_{O,50}$ 2. $\mathcal{R}_{O,-40}$ 3. $\mathcal{R}_{O,-90}$ 4. $\mathcal{R}_{O,400}$ 5. $\mathcal{R}_{O,-180}$

In the diagram for Exercises 6–11, O is the center of equilateral $\triangle PST$. State the images of points P, S, and T for each rotation.



Exs. 6-11

6. $\mathcal{R}_{O,120}$ 7. $\mathcal{R}_{O,-120}$ 8. $\mathcal{R}_{O,360}$

Name each image point.

9. $\mathcal{R}_{T,60}(S)$ 10. $\mathcal{R}_{T,-60}(P)$ 11. $\mathcal{R}_{O,240}(P)$

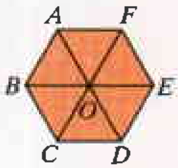
12. Draw a coordinate grid on the chalkboard. Plot the origin and $A(4, 1)$. Give the coordinates of **(a)** $H_O(A)$, **(b)** $\mathcal{R}_{O,90}(A)$, and **(c)** $\mathcal{R}_{O,-90}(A)$.
13. Repeat Exercise 12 if A has coordinates $(-3, 5)$.
14. Is congruence invariant under a half-turn mapping? Explain.
15. Read each expression aloud.
    a. $R_k(A) = A'$
    b. $H_O:(-2, 0) \rightarrow (2, 0)$
    c. $T:(x, y) \rightarrow (x - 1, y + 3)$
    d. $\mathcal{R}_{P,10}$

Written Exercises

State another name for each rotation.

A 1. $\mathcal{R}_{O, 80}$ 2. $\mathcal{R}_{O, -15}$ 3. $\mathcal{R}_{A, 450}$ 4. $\mathcal{R}_{B, -720}$ 5. H_O

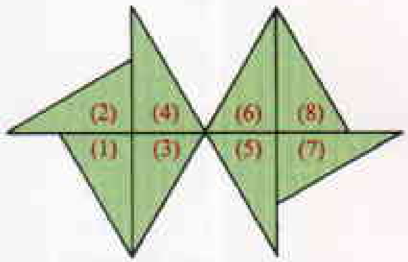
The diagonals of regular hexagon *ABCDEF* form six equilateral triangles as shown. Complete each statement below.



Exs. 6–14

6. $\mathcal{R}_{O, 60}: E \to$ ___?___
7. $\mathcal{R}_{O, -60}: D \to$ ___?___
8. $\mathcal{R}_{O, 120}: F \to$ ___?___
9. $\mathcal{R}_{D, 60}:$ ___?___ $\to O$
10. $\mathcal{R}_{B, -60}(O) =$ ___?___
11. $H_O(A) =$ ___?___
12. A reflection in $\overleftrightarrow{FC}$ maps B to ___?___ and D to ___?___.
13. If k is the perpendicular bisector of $\overline{FE}$, then $R_k(A) =$ ___?___.
14. If a translation maps A to B, then it also maps O to ___?___ and E to ___?___.

State whether the specified triangle is mapped to the other triangle by a reflection, translation, rotation, or half-turn.



Exs. 15–24

15. $\triangle(1)$ to $\triangle(2)$
16. $\triangle(1)$ to $\triangle(3)$
17. $\triangle(1)$ to $\triangle(4)$
18. $\triangle(1)$ to $\triangle(5)$
19. $\triangle(2)$ to $\triangle(4)$
20. $\triangle(2)$ to $\triangle(7)$
21. $\triangle(4)$ to $\triangle(6)$
22. $\triangle(4)$ to $\triangle(8)$

B 23. In the diagram at the right there is a glide reflection that maps triangle (1) to triangle (___?___).

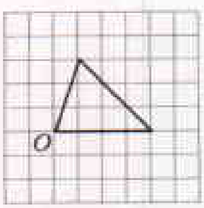
24. Name another pair of triangles for which one triangle is mapped to another by a glide reflection.

25. Which of the following properties are invariant under a half-turn?
 a. distance **b.** angle measure **c.** area **d.** orientation

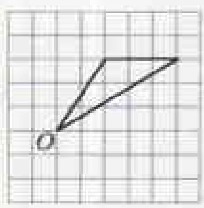
26. Which of the properties listed in Exercise 25 are invariant under the rotation $\mathcal{R}_{O, 90}$?

Copy the figure on graph paper. Draw the image by the specified rotation.

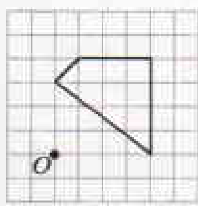
27. $\mathcal{R}_{O, 90}$



28. $\mathcal{R}_{O, -90}$



29. H_O



30. If $H_C: (1, 1) \to (7, 3)$, find the coordinates of C.

31. A rotation maps A to A' and B to B'. Construct the center of the rotation. (*Hint*: If the center is O, then $OA = OA'$ and $OB = OB'$.)

32. **a.** Draw a coordinate grid with origin O and plot the points $A(0, 3)$ and $B(4, 1)$.

b. Plot A' and B', the images of A and B by $\mathcal{R}_{O, 90}$.

c. Compare the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. What does this tell you about these lines?

d. Without using the distance formula, you know that $A'B' = AB$. State the theorem that tells you this.

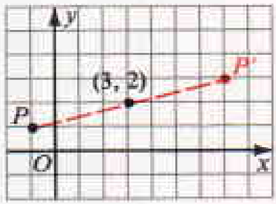
e. What reason supports the conclusion that $\triangle AOB$ and $\triangle A'OB'$ have the same area?

f. Use your graph to find the image of (x, y) by $\mathcal{R}_{O, 90}$.

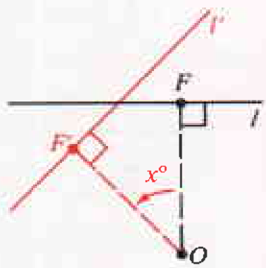
33. Repeat Exercise 32 using $\mathcal{R}_{O, 270}$.

34. A half-turn about $(3, 2)$ maps P to P'. Where does this half-turn map the following points?

a. P' **b.** $(0, 0)$ **c.** $(3, 0)$
d. $(1, 4)$ **e.** $(-2, 1)$ **f.** (x, y)



35. The rotation $\mathcal{R}_{O, x}$ maps line l to line l'. (You can think of rotating $\overline{OF}$, the perpendicular from O to l, through $x°$. Its image will be $\overline{OF'}$.) Show that one of the angles between l and l' has measure x.

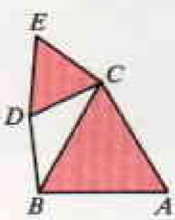


36. $\triangle ABC$ and $\triangle DCE$ are equilateral.

a. What rotation maps A to B and D to E?

b. Why does $AD = BE$?

c. Find the measure of an acute angle between $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$. (*Hint*: See Exercise 35.)

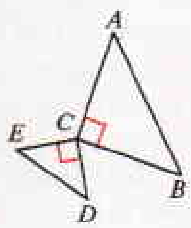


37. $\triangle ABC$ and $\triangle DEC$ are isosceles right triangles.

a. What rotation maps B to A and E to D?

b. Why does $AD = BE$?

c. Explain why $\overline{AD} \perp \overline{BE}$. (*Hint*: See Exercise 35.)



C 38. Given: Parallel lines l and k and point A.

a. Construct an equilateral $\triangle ABC$ with B on k and C on l using the following method.
Step 1. Rotate l through 60° about A and let B be the point on k where the image of l intersects k. (The diagram for Exercise 35 may be helpful in rotating l.)
Step 2. Let point C on l be the preimage of B.

b. Explain why $\triangle ABC$ is equilateral.

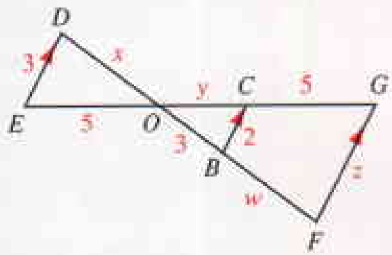
c. Are there other equilateral triangles with vertices at A and on l and k?

Exs. 38, 39

39. Given the figure for Exercise 38, construct a square $AXYZ$ with X on k and Z on l.

Mixed Review Exercises

Given $\overline{ED} \parallel \overline{BC} \parallel \overline{FG}$. Complete the statements.



1. $\triangle OBC$ is similar to $\triangle$ __?__ and $\triangle$ __?__.
2. The scale factor of $\triangle OBC$ to $\triangle ODE$ is __?__.
3. Find the values of x, y, z, and w.
4. The scale factor of $\triangle ODE$ to $\triangle OFG$ is __?__.
5. The ratio of the areas of $\triangle OBC$ and $\triangle ODE$ is __?__.
6. The ratio of the areas of $\triangle ODE$ and $\triangle OFG$ is __?__.
7. The ratio of the areas of $\triangle OBC$ and $\triangle OFG$ is __?__.

14-5 *Dilations*

Reflections, translations, glide reflections, and rotations are isometries, or *congruence* mappings. In this section we consider a transformation related to *similarity* rather than congruence. It is called a **dilation.** The dilation $D_{O,k}$ has *center O* and nonzero *scale factor k*. $D_{O,k}$ maps any point P to a point P' determined as follows:

(1) If $k > 0$, P' lies on $\overrightarrow{OP}$ and $OP' = k \cdot OP$.

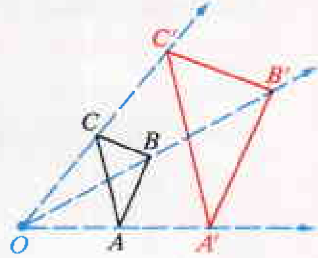
(2) If $k < 0$, P' lies on the ray opposite $\overrightarrow{OP}$ and $OP' = |k| \cdot OP$.

(3) The center O is its own image.

If $|k| > 1$, the dilation is called an **expansion.**
If $|k| < 1$, the dilation is called a **contraction.**

A developing leaf undergoes an expansion, keeping approximately the same shape as it grows in size.

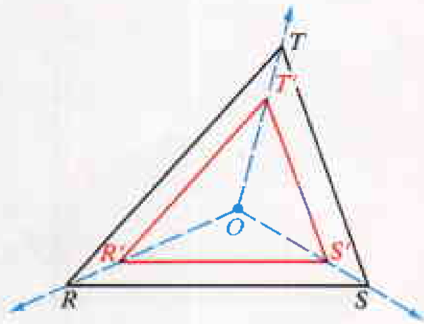
Example 1 Find the image of $\triangle ABC$ under the expansion $D_{O,2}$.



Solution $D_{O,2}: \triangle ABC \rightarrow \triangle A'B'C'$

$OA' = 2 \cdot OA$
$OB' = 2 \cdot OB$
$OC' = 2 \cdot OC$

Example 2 Find the image of $\triangle RST$ under the contraction $D_{O,\frac{2}{3}}$.

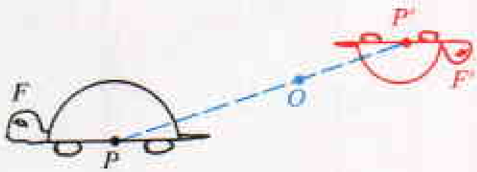


Solution $D_{O,\frac{2}{3}}: \triangle RST \rightarrow \triangle R'S'T'$

$OR' = \frac{2}{3} \cdot OR$
$OS' = \frac{2}{3} \cdot OS$
$OT' = \frac{2}{3} \cdot OT$

In the examples above, can you prove that the two triangles are similar? How are the areas of each pair of triangles related?

Example 3 Find the image of figure F under the contraction $D_{O,-\frac{1}{2}}$.



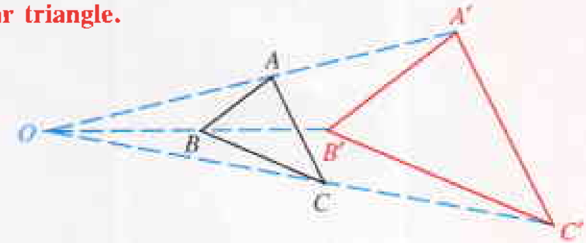
Solution $D_{O,-\frac{1}{2}}$: figure $F \rightarrow$ figure F'
$\overrightarrow{OP}$ is opposite to $\overrightarrow{OP'}$.
$OP' = \left|-\frac{1}{2}\right| \cdot OP = \frac{1}{2} \cdot OP$

If the scale factor in Example 3 was -1 instead of $-\frac{1}{2}$, the figure F' would be congruent to the figure F, and the transformation would be an isometry, equivalent to a half-turn. In general, however, as these examples illustrate, dilations do not preserve distance. Therefore a dilation is not an isometry (unless $k = 1$ or $k = -1$).

But a dilation always maps any geometric figure to a similar figure. In the examples above, $\triangle ABC \sim \triangle A'B'C'$, $\triangle RST \sim \triangle R'S'T'$ and the figure F is similar to the figure F'. For this reason, a dilation is an example of a **similarity mapping.**

Theorem 14-5

A dilation maps any triangle to a similar triangle.



Given: $D_{O,k}: \triangle ABC \to \triangle A'B'C'$

Prove: $\triangle ABC \sim \triangle A'B'C'$

Key steps of proof:

1. $OA' = |k| \cdot OA$, $OB' = |k| \cdot OB$ (Definition of dilation)
2. $\triangle OAB \sim \triangle OA'B'$ (SAS Similarity Theorem)
3. $\dfrac{A'B'}{AB} = \dfrac{OA'}{OA} = |k|$ (Corr. sides of ~ $\triangle$s are in proportion.)
4. Similarly, $\dfrac{B'C'}{BC} = \dfrac{A'C'}{AC} = |k|$ (Repeat Steps 1–3 for $\triangle OBC$ and $\triangle OB'C'$ and for $\triangle OAC$ and $\triangle OA'C'$.)
5. $\triangle ABC \sim \triangle A'B'C'$ (SSS Similarity Theorem)

Corollary 1

A dilation maps an angle to a congruent angle.

Corollary 2

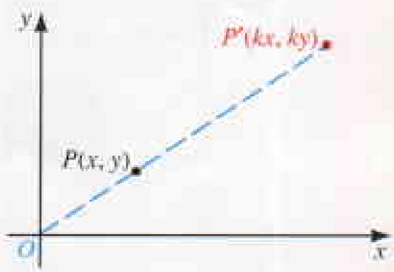
A dilation $D_{O,k}$ maps any segment to a parallel segment $|k|$ times as long.

Corollary 3

A dilation $D_{O,k}$ maps any polygon to a similar polygon whose area is k^2 times as large.

The diagram for the theorem above shows the case in which $k > 0$. You should draw the diagram for $k < 0$ and convince yourself that the proof is the same.

Theorem 14-5 can also be proved by using coordinates (see Exercise 28). To do this, you set the center of dilation at the origin, and describe $D_{O,k}$ in terms of coordinates by writing $D_{O,k}: (x, y) \to (kx, ky)$. You can see that this description satisfies the definition of a dilation because O, P, and P' are collinear (use slopes) and $OP' = |k| \cdot OP$ (use the distance formula).

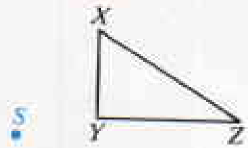


Classroom Exercises

Sketch each triangle on the chalkboard. Then sketch its image under the given dilation.

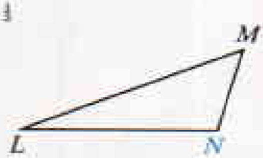
1. $D_{O,3}$

2. $D_{S,\frac{1}{2}}$



3. $D_{E,-2}$

4. $D_{N,-\frac{1}{3}}$

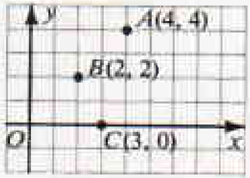


5. Find the coordinates of the images of points A, B, and C under the dilation $D_{O,2}$.

6. Find the image of (x, y) under $D_{O,2}$.

7. What dilation with center O maps A to B?

8. What dilation with center O maps C to the point $(-6, 0)$?



Exs. 5–10

9. Find the coordinates of the image of A under $D_{B,2}$.

10. Find the coordinates of the image of B under $D_{C,3}$.

11. Match each scale factor in the first column with the name of the corresponding dilation in the second column.

Scale factor	*Transformation*
$\frac{2}{5}$	Half-turn
-4	Contraction
-1	Expansion

12. Describe the dilation $D_{O,1}$.

13. If $\odot S$ has radius 4, describe the image of $\odot S$ under $D_{S,5}$ and under $D_{S,-1}$.

14. If point A is on line k, what is the image of line k under $D_{A,2}$?

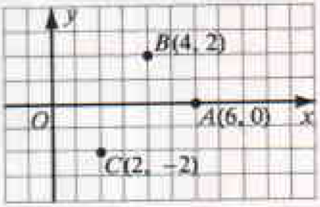
15. The dilation $D_{O,3}$ maps P to P' and Q to Q'.

a. If $OQ = 2$, find OQ'.
b. If $PQ = 7$, find $P'Q'$.
c. If $PP' = 10$, find OP.

16. Explain how Corollary 1 follows from Theorem 14-5.

17. Explain how Corollary 3 follows from Corollaries 1 and 2.

Written Exercises



Find the coordinates of the images of A, B, and C by the given dilation.

A

1. $D_{O,2}$
2. $D_{O,3}$
3. $D_{O,\frac{1}{2}}$
4. $D_{O,-\frac{1}{2}}$
5. $D_{O,-2}$
6. $D_{O,1}$
7. $D_{A,-\frac{1}{2}}$
8. $D_{A,2}$

A dilation with the origin, O, as center maps the given point to the image point named. Find the scale factor of the dilation. Is the dilation an expansion or a contraction?

9. $(2, 0) \rightarrow (8, 0)$
10. $(2, 3) \rightarrow (4, 6)$
11. $(3, 9) \rightarrow (1, 3)$
12. $(4, 10) \rightarrow (-2, -5)$
13. $(0, \frac{1}{6}) \rightarrow (0, \frac{2}{3})$
14. $(-6, 2) \rightarrow (18, -6)$

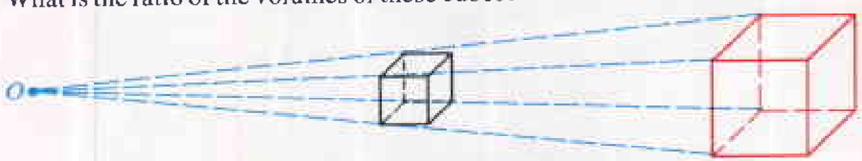
B

15. Which of the following properties are invariant under any dilation?
    a. distance b. angle measure c. area d. orientation
16. Is parallelism invariant under a dilation? (*Hint*: See Exercise 23 on page 581.)
17. If A', B', C', and D' are the images of any four points A, B, C, and D, then we say the ratio of distances is invariant under the transformation if $\frac{AB}{CD} = \frac{A'B'}{C'D'}$. For which of the following transformations is the ratio of distances invariant?
    a. reflection b. rotation c. dilation

Graph quad. $PQRS$ and its image by the dilation given. Find the ratio of the perimeters and the ratio of the areas of the two quadrilaterals.

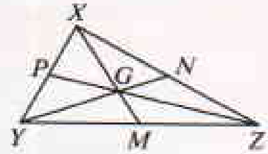
18.	$P(-1, 1)$	$Q(0, -1)$	$R(4, 0)$	$S(2, 2)$	$D_{O,3}$
19.	$P(12, 0)$	$Q(0, 15)$	$R(-9, 6)$	$S(3, -9)$	$D_{O,\frac{2}{3}}$
20.	$P(3, 0)$	$Q(3, 4)$	$R(6, 6)$	$S(5, -1)$	$D_{O,-2}$
21.	$P(-2, -2)$	$Q(0, 0)$	$R(4, 0)$	$S(6, -2)$	$D_{O,-\frac{1}{2}}$

22. The diagram illustrates a dilation of three-dimensional space. $D_{O,2}$ maps the smaller cube to the larger cube.
    a. What is the ratio of the surface areas of these cubes?
    b. What is the ratio of the volumes of these cubes?



23. A dilation with scale factor $\frac{3}{4}$ maps a sphere with center C to a concentric sphere.
    a. What is the ratio of the surface areas of these spheres?
    b. What is the ratio of the volumes of these spheres?

24. G is the intersection of the medians of $\triangle XYZ$. Complete the following statements. (*Hint*: Use Theorem 10-4 on page 387.)



a. $\frac{XG}{XM} = \underline{\ ?\ }$ **b.** $\frac{GM}{GX} = \underline{\ ?\ }$

c. What dilation maps X to M?

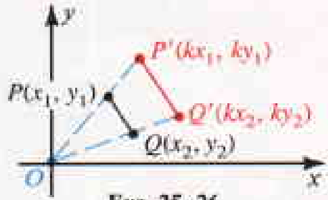
d. What is the image under this dilation of Y? of Z?

25. $D_{O,k}$ maps $\overline{PQ}$ to $\overline{P'Q'}$.

a. Show that the slopes of $\overline{PQ}$ and $\overline{P'Q'}$ are equal.

b. Part (a) proves that $\overline{PQ}$ and $\overline{P'Q'}$ are $\underline{\ ?\ }$.

C **26.** Use the distance formula to show that
$P'Q' = |k|\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |k| \cdot PQ$.



Exs. 25, 26

27. A dilation with center (a, b) and scale factor k maps $A(3, 4)$ to $A'(1, 8)$, and $B(3, 2)$ to $B'(1, 2)$. Find the coordinates of the center (a, b) and the value of k.

28. Prove Theorem 14-5 using the coordinate definition of a dilation, $D_{O,k}:(x, y) \to (kx, ky)$. (*Hint*: Let A, B, and C have coordinates (p, q), (r, s), and (t, u) respectively.)

Self-Test 1

1. Define an isometry.

2. If $f(x) = 3x - 7$, find the image of 2 and the preimage of 2.

3. If $T:(x, y) \to (x + 1, y - 2)$, find the image and preimage of the origin.

4. Find the image of (3, 5) when reflected in each line.

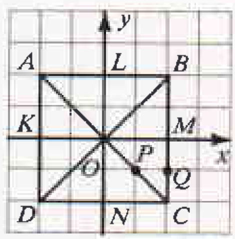
a. the x-axis **b.** the y-axis **c.** the line $y = x$.

5. A dilation with scale factor 3 maps $\triangle ABC$ to $\triangle A'B'C'$. Which of the following are true?

a. $\overline{AB} \parallel \overline{A'B'}$ **b.** $\frac{A'B'}{AB} = 3$ **c.** $\frac{\text{area of } \triangle A'B'C'}{\text{area of } \triangle ABC} = 3$

6. Give two other names for the rotation $\mathcal{R}_{O,-30}$.

Complete. R_x **and** R_y **denote reflections in the** x**- and** y**-axes, respectively.**



7. $R_y:A \to \underline{\ ?\ }$ **8.** $R_x:B \to \underline{\ ?\ }$

9. $R_x:\overline{DC} \to \underline{\ ?\ }$ **10.** $R_y:\underline{\ ?\ } \to \overline{OA}$

11. $H_O:K \to \underline{\ ?\ }$ **12.** $H_O:\underline{\ ?\ } \to \overline{CO}$

13. $\mathcal{R}_{O,90}$ maps M to $\underline{\ ?\ }$. **14.** $\mathcal{R}_{O,-90}$ maps $\triangle MCO$ to $\triangle \underline{\ ?\ }$.

15. $D_{O,2}$ maps P to $\underline{\ ?\ }$. **16.** $D_{M,-\frac{1}{2}}$ maps B to $\underline{\ ?\ }$.

17. A translation that maps A to L maps N to $\underline{\ ?\ }$.

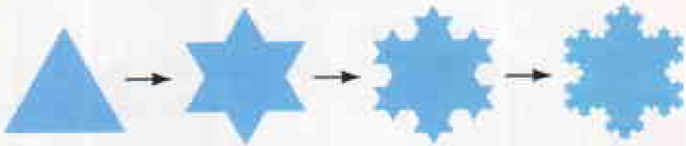
18. The glide reflection in $\overleftrightarrow{BD}$ that maps K to M maps N to $\underline{\ ?\ }$.

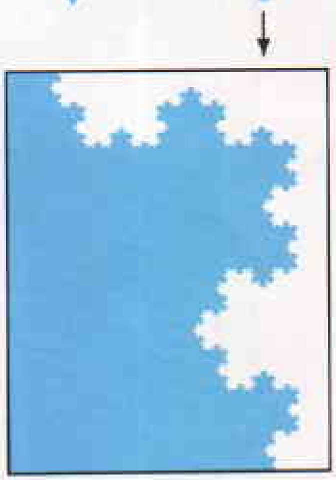
Career

Computer Animation Programmer

One problem computer animation programmers have encountered is how to produce natural-looking landscapes. The structures of trees, mountains, clouds, and coastlines are complex. To create them in a computer landscape can require storing a great deal of information. Also, since animations often include moving through space, data about the landscape features needs to be provided at many levels of detail. (If you specified the appearance of a mountain from only one viewpoint, say, then "zooming in" for a close-up would reveal that details are missing, a problem known as *loss of resolution.*)

One new approach involves using fractals. A *fractal* is a complex shape that looks more or less the same at all magnifications. Fractals are made by following simple rules, called *algorithms.* The snowflake shape in the diagram at right is an example. Its algorithm is: Start with an equilateral triangle. Divide each side of the polygon in thirds; add an equilateral triangle to each center third; repeat. No matter how much you magnify a piece of this polygon, it will retain the overall pattern and complexity of the original. When you "zoom in" on a fractal shape, there is no loss of resolution.





Computer programmers are taking advantage of this property of fractals to approximate many items in nature, such as the mountains in the photograph above. Programming a computer to follow these algorithms uses less memory than specifying the exact shape of each element from many different viewpoints and distances.

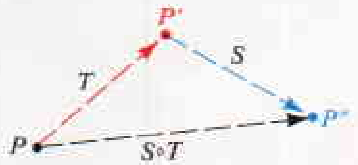
Composition and Symmetry

Objectives

1. Locate the images of figures by composites of mappings.
2. Recognize and use the terms *identity* and *inverse* in relation to mappings.
3. Describe the symmetry of figures and solids.

14-6 *Composites of Mappings*

Suppose a transformation T maps point P to P' and then a transformation S maps P' to P''. Then T and S can be combined to produce a new transformation that maps P directly to P''. This new transformation is called the **composite** of S and T and is written $S \circ T$. Notice in the diagram that $P'' = S(P') = S(T(P)) = (S \circ T)(P)$.



We reduce the number of parentheses needed to indicate that the composite of S and T maps P to P'' by writing $S \circ T: P \rightarrow P''$. Notice that T, the transformation that is applied first, is written closer to P, and S, the transformation that is applied second, is written farther from P. For this reason, the composite $S \circ T$ is often read "S after T," or "T followed by S."

The operation that combines two mappings (or functions) to produce the composite mapping (or composite function) is called *composition*. We shall see that composition has many characteristics similar to multiplication, but there is one important exception. For multiplication, it makes no difference in which order you multiply two numbers. For composition, however, the order of the mappings or functions usually *does* make a difference. Examples 1 and 2 illustrate this.

Example 1 If $f(x) = x^2$ and $g(x) = 2x$, find **(a)** $(g \circ f)(x)$ and **(b)** $(f \circ g)(x)$.

Solution

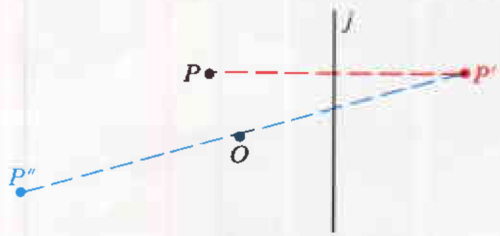
a. $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$

b. $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2$, or $4x^2$

In mapping notation, we could write that $g \circ f: x \rightarrow 2x^2$ and $f \circ g: x \rightarrow 4x^2$. Note that since $2x^2 \neq 4x^2$, $g \circ f \neq f \circ g$.

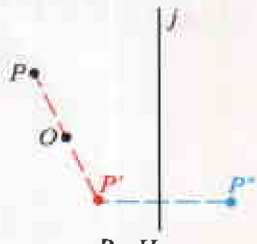
Example 2 Show that $H_O \circ R_j \neq R_j \circ H_O$.

Solution Study the two diagrams below.



$H_O \circ R_j$

Here R_j, the reflection of P in line j, is carried out first, mapping P to P'. Then H_O maps P' to P''. Thus P'' is the image of P under the composite $H_O \circ R_j$.



$R_j \circ H_O$

With the order changed in the composite, the half-turn is carried out first, followed by the reflection in line j. The image point P'' is now in a different place.

Notice that the two composites map P to different image points, so the composites are not equal.

Example 2 shows that the order in a composite of transformations can be very important, but this is not always true. For example, if S and T are two translations, then order is not important, since $S \circ T = T \circ S$ (see Exercise 10).

Example 2 above shows the effect of a composite of mappings on a single point P. The diagram below shows a composite of reflections acting on a whole figure, F. F is reflected in line j to F', and F' is reflected in line k to F''. Thus $R_k \circ R_j$ maps F to F''. Again notice that the first reflection, R_j, is written on the right.

The final image F'' is the same size and shape as F. Also, F'' is the image of F under a translation. This illustrates our next two theorems. First, the composite of any two isometries is an isometry. Second, the composite of reflections in two parallel lines is a translation.

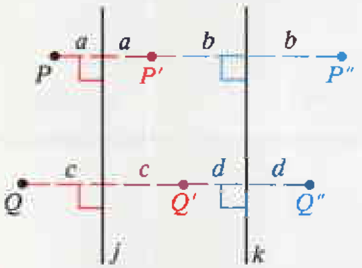
Theorem 14-6

The composite of two isometries is an isometry.

Theorem 14-7

A composite of reflections in two parallel lines is a translation. The translation glides all points through twice the distance from the first line of reflection to the second.

Although we will not present a formal proof of Theorem 14-7, the following argument should convince you that it is true. Assume that $j \parallel k$ and that R_j maps P to P' and Q to Q', and that R_k maps P' to P'' and Q' to Q''. To show that the composite $R_k \circ R_j$ is a translation we will demonstrate that $PP'' = QQ''$ and that $\overleftrightarrow{PP''}$ and $\overleftrightarrow{QQ''}$ are parallel.



The letters a, b, c, and d in the diagram label pairs of distances that are equal according to the definition of a reflection. P, P', and P'' are collinear and

$$PP'' = 2a + 2b = 2(a + b)$$

Similarly,

$$QQ'' = 2c + 2d = 2(c + d)$$

But $(a + b) = (c + d)$, since by Theorem 5-8, the distance between the parallel lines j and k is constant. Therefore $PP'' = QQ'' =$ twice the distance from j to k.

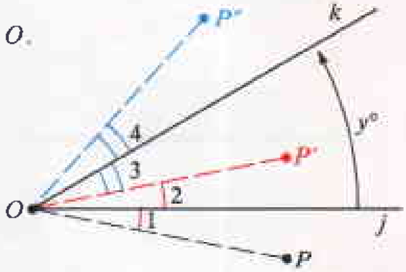
That $\overleftrightarrow{PP''}$ and $\overleftrightarrow{QQ''}$ are parallel follows from the fact that both lines are perpendicular to j and k. Theorem 3-7 guarantees that if two lines in a plane are perpendicular to the same line, then the two lines are parallel.

You should make diagrams for the case when P is on j or k, when P is located between j and k, and when P is to the right of k. Convince yourself that $PP'' = 2(a + b)$ in these cases also. In every case, the glide is perpendicular to j and k and goes in the direction from j to k (that is, from the first line of reflection toward the second line of reflection).

Theorem 14-7 shows that when lines j and k are parallel, $R_k \circ R_j$ translates points through twice the distance between the lines. If lines j and k intersect, $R_k \circ R_j$ rotates points through twice the measure of the angle between the lines. This is our next theorem.

Theorem 14-8

A composite of reflections in two intersecting lines is a rotation about the point of intersection of the two lines. The measure of the angle of rotation is twice the measure of the angle from the first line of reflection to the second.



Given: j intersects k, forming an angle of measure y at O.

Prove: $R_k \circ R_j = \mathcal{R}_{O,\,2y}$

Proof:

The diagram shows an arbitrary point P and its image P' by reflection in j. The image of P' by reflection in k is P''. According to the definition of a rotation we must prove that $OP = OP''$ and $m\angle POP'' = 2y$.

R_j and R_k are isometries, so they preserve both distance and angle measure. Therefore $OP = OP'$, $OP' = OP''$, $m\angle 1 = m\angle 2$, and $m\angle 3 = m\angle 4$. Thus $OP = OP''$ and the measure of the angle of rotation equals

$$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 2m\angle 2 + 2m\angle 3 = 2y.$$

Corollary

A composite of reflections in perpendicular lines is a half-turn about the point where the lines intersect.

Classroom Exercises

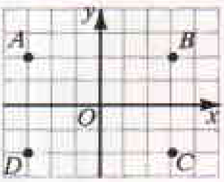
1. If $f(x) = x + 1$ and $g(x) = 3x$, find the following.

a. $f(4)$ **b.** $(g \circ f)(4)$ **c.** $(g \circ f)(x)$

d. $g(2)$ **e.** $(f \circ g)(2)$ **f.** $(f \circ g)(x)$

2. Repeat Exercise 1 if $f: x \rightarrow \sqrt{x}$ and $g: x \rightarrow x + 7$.

Complete the following. R_x and R_y are reflections in the x- and y-axes.



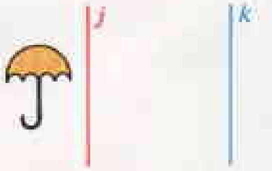
3. $R_x \circ R_y : A \rightarrow$ _?_ **4.** $R_x \circ R_y : D \rightarrow$ _?_

5. $H_O \circ R_y : B \rightarrow$ _?_ **6.** $R_y \circ H_O : B \rightarrow$ _?_

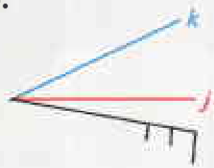
7. $H_O \circ H_O : A \rightarrow$ _?_ **8.** $R_y \circ R_y : C \rightarrow$ _?_

Copy the figure on the chalkboard and find its image by $R_k \circ R_j$. Then copy the figure again and find its image by $R_j \circ R_k$.

9.



10.



11. Prove Theorem 14-6. (*Hint*: Let S and T be isometries. Consider a $\overline{PQ}$ under $S \circ T$.)

12. Explain how the Corollary follows from Theorem 14-8.

Written Exercises

A

1. If $f(x) = x^2$ and $g(x) = 2x - 7$, evaluate the following.
a. $(g \circ f)(2)$ **b.** $(g \circ f)(x)$ **c.** $(f \circ g)(2)$ **d.** $(f \circ g)(x)$

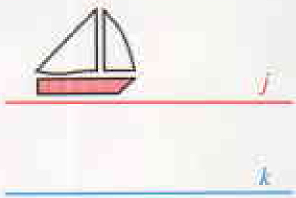
2. Repeat Exercise 1 if $f(x) = 3x + 1$ and $g(x) = x - 9$.

3. If $h:x \to \frac{x+1}{2}$ and $k:x \to x^3$, complete the following.
a. $k \circ h:3 \to$ _?_ **b.** $k \circ h:5 \to$ _?_ **c.** $k \circ h:x \to$ _?_
d. $h \circ k:3 \to$ _?_ **e.** $h \circ k:5 \to$ _?_ **f.** $h \circ k:x \to$ _?_

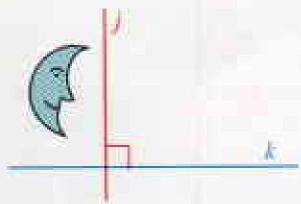
4. Repeat Exercise 3 if $h:x \to x^2 - 1$ and $k:x \to 2x + 7$.

Copy each figure and find its image under $R_k \circ R_j$. Then copy the figure again and find its image under $R_j \circ R_k$.

5.



6.



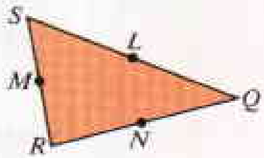
Copy each figure twice and show the image of the red flag under each of the composites given.

7. a. $H_B \circ H_A$
b. $H_A \circ H_B$

8. a. $R_j \circ H_C$
b. $H_C \circ R_j$

9. a. $H_E \circ D_{E,\frac{1}{3}}$
b. $D_{E,\frac{1}{3}} \circ H_E$

10. Given $A(4, 1)$, $B(1, 5)$, and $C(0, 1)$. S and T are translations. $S:(x, y) \rightarrow (x + 1, y + 4)$ and $T:(x, y) \rightarrow (x + 3, y - 1)$. Draw $\triangle ABC$ and its images under $S \circ T$ and $T \circ S$.
   a. Does $S \circ T$ appear to be a translation?
   b. Is $S \circ T$ equal to $T \circ S$?
   c. $S \circ T:(x, y) \rightarrow$ (_?_, _?_) and $T \circ S:(x, y) \rightarrow$ (_?_, _?_)



Exs. 11, 12

11. L, M, and N are midpoints of the sides of $\triangle QRS$.
   a. $H_N \circ H_M:S \rightarrow$ _?_   **b.** $H_M \circ H_N:Q \rightarrow$ _?_
   c. $D_{S,\frac{1}{2}} \circ H_N:Q \rightarrow$ _?_   **d.** $H_N \circ D_{S,2}:M \rightarrow$ _?_
   e. $H_L \circ H_M \circ H_N:Q \rightarrow$ _?_

B **12.** If T is a translation that maps R to N, then:
   a. $T:M \rightarrow$ _?_   **b.** $T \circ D_{S,\frac{1}{2}}:R \rightarrow$ _?_   **c.** $T \circ T:R \rightarrow$ _?_

In Exercises 13–16 tell which of the following properties are invariant under the given transformation.
a. distance   b. angle measure   c. area   d. orientation

13. The composite of a reflection and a dilation
14. The composite of two reflections
15. The composite of a rotation and a translation
16. The composite of two dilations

For each exercise draw a grid and find the coordinates of the image point. *O* is the origin and *A* is the point (3, 1). R_x and R_y are reflections in the *x*- and *y*-axes.

17. $R_x \circ R_y:(3, 1) \rightarrow$ (_?_, _?_)
18. $R_y \circ H_O:(1, -2) \rightarrow$ (_?_, _?_)
19. $H_A \circ H_O:(3, 0) \rightarrow$ (_?_, _?_)
20. $H_O \circ H_A:(1, 1) \rightarrow$ (_?_, _?_)
21. $R_x \circ D_{O,2}:(2, 4) \rightarrow$ (_?_, _?_)
22. $\mathcal{R}_{O,90} \circ R_y:(-2, 1) \rightarrow$ (_?_, _?_)
23. $\mathcal{R}_{A,90} \circ \mathcal{R}_{O,-90}:(-1, -1) \rightarrow$ (_?_, _?_)
24. $D_{O,-\frac{1}{3}} \circ D_{A,4}:(3, 0) \rightarrow$ (_?_, _?_)

25. Let R_l be a reflection in the line $y = x$ and R_y be a reflection in the y-axis. Draw a grid and label the origin O.
   a. Plot the point $P(5, 2)$ and its image Q under the mapping $R_y \circ R_l$.
   b. According to Theorem 14-8, $m\angle POQ =$ _?_.
   c. Use the slopes of $\overline{OP}$ and $\overline{OQ}$ to verify that $\overline{OP} \perp \overline{OQ}$.
   d. Find the images of (x, y) under $R_y \circ R_l$ and $R_l \circ R_y$.

26. Let R_k be a reflection in the line $y = -x$ and R_x be a reflection in the x-axis.
   a. Plot $P(-6, -2)$ and its image Q under the mapping $R_k \circ R_x$.
   b. Use slopes to show that $m\angle POQ = 90$ where O is the origin. (Do you see that this result agrees with Theorem 14-8?)
   c. Find the images of (x, y) under $R_k \circ R_x$ and $R_x \circ R_k$.

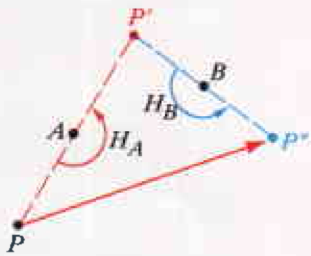
C **27.** Explain how you would construct line j so that $R_k \circ R_j:A \rightarrow B$.

28. The figure shows that $H_B \circ H_A: P \to P''$.

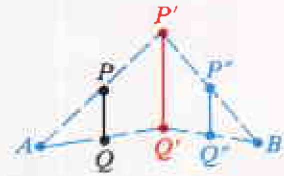
a. Copy the figure and verify by measuring that $PP'' = 2 \cdot AB$. What theorem about the midpoints of the sides of a triangle does this suggest?

b. Choose another point Q and carefully locate Q'', the image of Q under $H_B \circ H_A$. Does $QQ'' = 2 \cdot AB$?

c. Measure PQ and $P''Q''$. Are they equal? What kind of transformation does $H_B \circ H_A$ appear to be?



29. $D_{A,2}: \overline{PQ} \to \overline{P'Q'}$ and $D_{B,\frac{1}{2}}: \overline{P'Q'} \to \overline{P''Q''}$. What kind of transformation is the composite $D_{B,\frac{1}{2}} \circ D_{A,2}$? Explain.



30. The point P is called a *fixed point* of the transformation T if $T: P \to P$.

a. How many fixed points does each of the following have: $\mathcal{R}_{O,90}$? R_y? $D_{O,3}$? the translation $T: (x, y) \to (x - 3, y + 2)$?

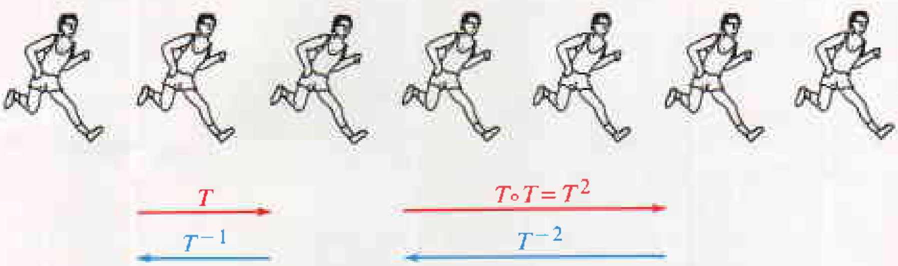
b. O is the origin and A is the point $(1, 0)$. Find the coordinates of a fixed point of the composite $D_{O,2} \circ D_{A,\frac{1}{4}}$.

14-7 *Inverses and the Identity*

Suppose that the pattern below continues indefinitely to both the left and the right. The translation T glides each runner one place to the right. The translation that glides each runner one place to the *left* is called the *inverse* of T, and is denoted T^{-1}. Notice that T followed by T^{-1} keeps *all* points fixed:

$$T^{-1} \circ T: P \to P$$

The composite $T \circ T$, also written $T \cdot T$, and usually denoted by T^2, glides each runner two places to the right.



The mapping that maps every point to itself is called the **identity** transformation I. The words "identity" and "inverse" are used for mappings in much the same way that they are used for numbers. In fact, the composite of two mappings is very much like the product of two numbers. For this reason, the composite $S \circ T$ is often called the **product** of S and T.

Relating Algebra and Geometry

For products of numbers	*For composites of mappings*
1 is the identity.	I is the identity.
$a \cdot 1 = a$ and $1 \cdot a = a$	$S \circ I = S$ and $I \circ S = S$
The inverse of a is written a^{-1}, or $\frac{1}{a}$.	The inverse of S is written S^{-1}.
$a \cdot a^{-1} = 1$ and $a^{-1} \cdot a = 1$	$S \circ S^{-1} = I$ and $S^{-1} \circ S = I$

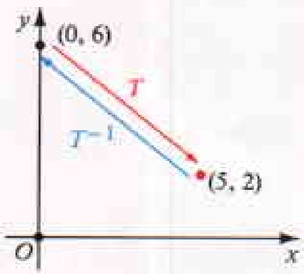
In general, the **inverse** of a transformation T is defined as the transformation S such that $S \circ T = I$. The inverses of some other transformations are illustrated below.

Example 1 Find the inverses of **(a)** translation $T:(x, y) \rightarrow (x + 5, y - 4)$, **(b)** rotation $\mathcal{R}_{O,\,x}$, and **(c)** dilation $D_{O,\,2}$.

Solution

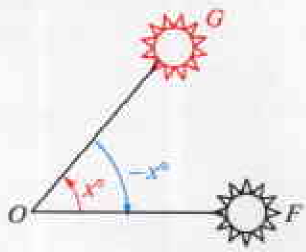
a. $T^{-1}:(x, y) \rightarrow (x - 5, y + 4)$

$T:(0, 6) \rightarrow (5, 2)$
$T^{-1}:(5, 2) \rightarrow (0, 6)$



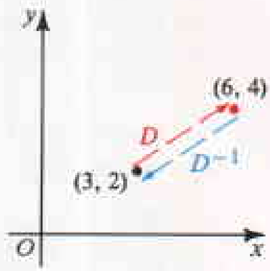
b. The inverse of $\mathcal{R}_{O,\,x}$ is $\mathcal{R}_{O,\,-x}$.

$\mathcal{R}_{O,\,x}:F \rightarrow G$
$\mathcal{R}_{O,\,-x}:G \rightarrow F$

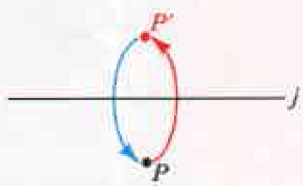


c. The inverse of $D_{O,\,2}$ is $D_{O,\,\frac{1}{2}}$.

$D_{O,\,2}:(3, 2) \rightarrow (6, 4)$
$D_{O,\,\frac{1}{2}}:(6, 4) \rightarrow (3, 2)$



Example 2 What is the inverse of R_j? (Refer to the diagram at right.)



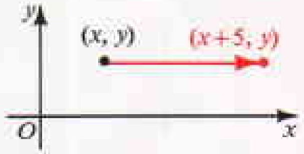
Solution Since $R_j \circ R_j = I$, the inverse of R_j is R_j itself. In symbols, $R_j^{-1} = R_j$. Do you see that the inverse of any reflection is that same reflection?

Classroom Exercises

The symbol 2^{-1} stands for the multiplicative inverse of 2, or $\frac{1}{2}$. Give the value of each of the following.

1. 3^{-1} **2.** 7^{-1} **3.** $(\frac{4}{5})^{-1}$ **4.** $(2^{-1})^{-1}$

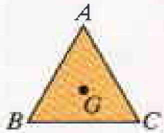
The translation T maps all points five units to the right. Describe each of the following transformations.



5. T^2 **6.** T^3 **7.** T^{-1}

8. T^{-2} **9.** $T \circ T^{-1}$ **10.** $(T^{-1})^{-1}$

The rotation $\mathcal{R}$ maps all points 120° about G, the center of equilateral $\triangle ABC$. Give the image of A under each rotation.



11. $\mathcal{R}$ **12.** $\mathcal{R}^2$ **13.** $\mathcal{R}^3$

14. $\mathcal{R}^6$ **15.** $\mathcal{R}^{-1}$ **16.** $\mathcal{R}^{-2}$

17. $\mathcal{R}^2 \circ \mathcal{R}^{-2}$ **18.** $\mathcal{R}^2 \circ \mathcal{R}^{-3}$ **19.** $\mathcal{R}^{100}$

20. What number is the identity for multiplication?

21. The product of any number t and the identity for multiplication is __?__.

22. The product of any transformation T and the identity is __?__.

23. State the inverse of each transformation.

a. R_l **b.** $\mathcal{R}_{O, 30}$ **c.** $T:(x, y) \to (x - 4, y + 1)$ **d.** $D_{O, -1}$

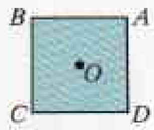
24. Name an important difference between products of numbers and products of transformations.

Written Exercises

Give the value of each of the following.

A **1.** 4^{-1} **2.** 9^{-1} **3.** $(\frac{2}{3})^{-1}$ **4.** $(5^{-1})^{-1}$

The rotation $\mathcal{R}$ maps all points 90° about O, the center of square $ABCD$. Give the image of A under each rotation.



5. $\mathcal{R}^2$ **6.** $\mathcal{R}^3$ **7.** $\mathcal{R}^4$

8. $\mathcal{R}^{-1}$ **9.** $\mathcal{R}^{-2}$ **10.** $\mathcal{R}^{-3}$

11. $\mathcal{R}^{-3} \circ \mathcal{R}^3$ **12.** $\mathcal{R}^5$ **13.** $\mathcal{R}^{50}$

Complete.

14. By definition, the identity mapping I maps every point P to __?__.

15. $H_O{}^2$ is the same as the mapping __?__.

16. The inverse of H_O is __?__.

17. $H_O{}^3$ is the same as the mapping __?__.

18. If $T:(x, y) \to (x + 2, y)$, then $T^2:(x, y) \to (\underline{\ ?\ }, \underline{\ ?\ })$.

19. If $T:(x, y) \to (x + 3, y - 4)$, then $T^2:(x, y) \to (\underline{\ ?\ }, \underline{\ ?\ })$.

20. If R_x is reflection in the x-axis, then $(R_x)^2:P \to \underline{\ ?\ }$.

In each exercise, a rule is given for a mapping S. Write the rule for S^{-1}.

B **21.** $S:(x, y) \to (x + 5, y + 2)$ **22.** $S:(x, y) \to (x - 3, y - 1)$

23. $S:(x, y) \to (3x, -\frac{1}{2}y)$ **24.** $S:(x, y) \to (\frac{1}{4}x, \frac{1}{4}y)$

25. $S:(x, y) \to (x - 4, 4y)$ **26.** $S:(x, y) \to (y, x)$

27. If $S:(x, y) \to (x + 12, y - 3)$, find a translation T such that $T^6 = S$.

28. Find a transformation S (other than the identity) for which $S^5 = I$.

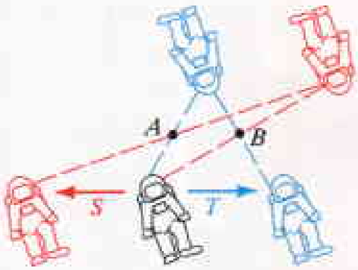
C **29.** **a.** j and k are vertical lines 1 unit apart. According to Theorem 14-7, $R_k \circ R_j$ and $R_j \circ R_k$ are both translations. Describe in words the distance and direction of each translation.

b. Show that $R_k \circ R_j$ and $R_j \circ R_k$ are inverses by showing that their composite is I. Note: Forming composites of transformations is an associative operation, so $(R_k \circ R_j) \circ (R_j \circ R_k) = R_k \circ (R_j \circ R_j) \circ R_k$.

30. The blue lines in the diagram illustrate the statement $H_B \circ H_A$ = translation T. The red lines show that $H_A \circ H_B$ = translation S.

a. How is translation S related to translation T?

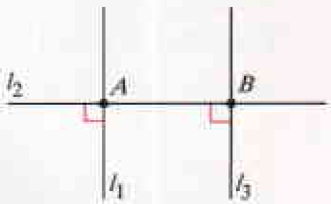
b. Prove your answer correct by showing that $(H_A \circ H_B) \circ (H_B \circ H_A) = I$. (*Hint*: See Exercise 29, part (b).)



31. Complete the proof by giving a reason for each step.

Given: $l_1 \perp l_2$; $l_3 \perp l_2$; R_1, R_2, and R_3 denote reflections in l_1, l_2, and l_3.

Prove: $H_B \circ H_A$ is a translation.



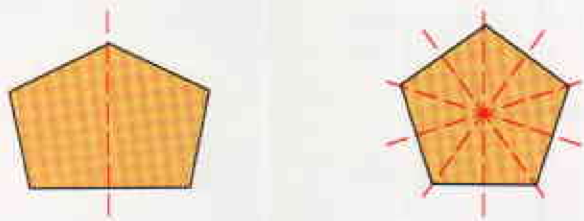
Proof:

Statements	Reasons
1. $H_A = R_2 \circ R_1$	1. ?
2. $H_B = R_3 \circ R_2$	2. ?
3. $H_B \circ H_A = (R_3 \circ R_2) \circ (R_2 \circ R_1)$	3. ?
4. $H_B \circ H_A = (R_3 \circ (R_2 \circ R_2)) \circ R_1$	4. Composition is associative.
5. $H_B \circ H_A = (R_3 \circ I) \circ R_1$	5. ?
6. $H_B \circ H_A = R_3 \circ R_1$	6. ?
7. $H_B \circ H_A$ is a translation.	7. ?

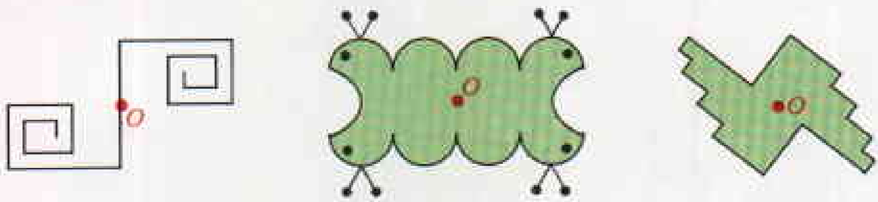
14-8 Symmetry in the Plane and in Space

A figure in the plane has **symmetry** if there is an isometry, other than the identity, that maps the figure onto itself. We call such an isometry a *symmetry* of the figure.

Both of the figures below have **line symmetry.** This means that for each figure there is a symmetry line k such that the reflection R_k maps the figure onto itself. The pentagon at the left has one symmetry line. The regular pentagon at the right has five symmetry lines.



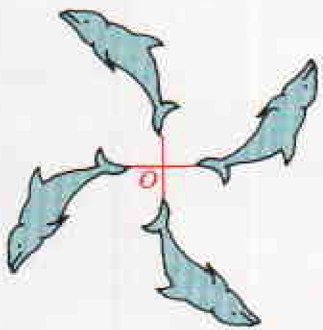
Each figure below has **point symmetry.** This means that for each figure there is a symmetry point O such that the half-turn H_O maps the figure onto itself.



Besides having a symmetry point, the middle figure above has a vertical symmetry line and a horizontal symmetry line.

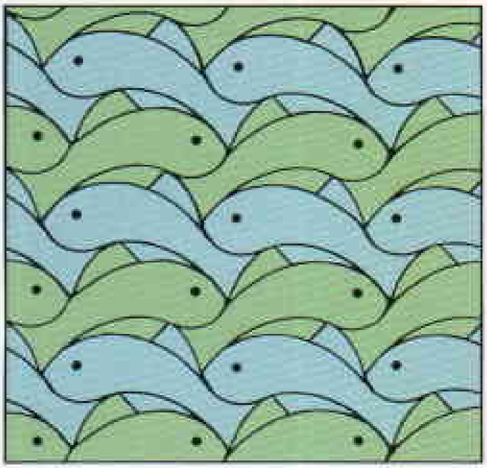
A third kind of symmetry is **rotational symmetry.** The figure below has the four rotational symmetries listed. Each symmetry has center O and rotates the figure onto itself. Note that 180° rotational symmetry is another name for point symmetry.

(1) 90° rotational symmetry: $\mathcal{R}_{O,\,90}$
(2) 180° rotational symmetry: $\mathcal{R}_{O,\,180}$ (or H_O)
(3) 270° rotational symmetry: $\mathcal{R}_{O,\,270}$
(4) 360° rotational symmetry: the identity I



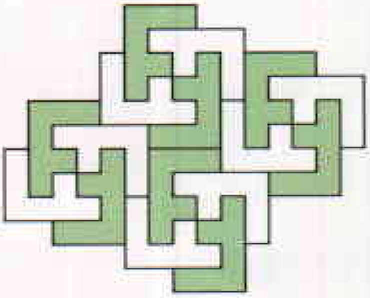
The identity mapping always maps a figure onto itself, and we usually include the identity when listing the symmetries of a figure. However, we do not call a figure *symmetric* if the identity is its only symmetry.

A figure can also have **translational symmetry** if there is a translation that maps the figure onto itself. For example, imagine that the design at the right extends in all directions to fill the plane. If you consider the distance between the eyes of adjacent blue fish as a unit, then a translation through one or more units right, left, up, or down maps the whole pattern onto itself. Do you see that you can also translate the pattern along diagonal lines?

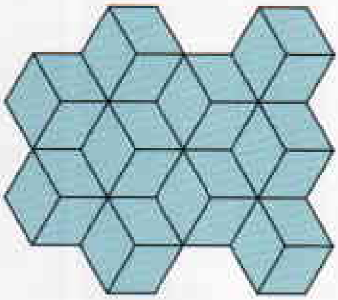


It is also possible to map the blue fish, which all face to the left, onto the right-facing green fish by translating the whole pattern a half unit up and then reflecting it in a vertical line. Thus, if we ignore color differences, the pattern has **glide reflection symmetry.**

A design like this pattern of fish, in which congruent copies of a figure completely fill the plane without overlapping, is called a *tessellation*. Tessellations can have any of the kinds of symmetry we have discussed. Here are two more examples.



A tessellation of the letter F. This pattern has point symmetry and translational symmetry.

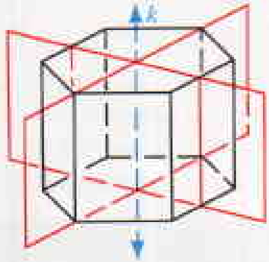


This tessellation has line, point, rotational, translational, and glide reflection symmetry.

Coloring a tessellation often changes its symmetries. For example, if the green were removed from the tessellation of the letter F, the pattern would also have $90°$ and $270°$ rotational symmetry.

A figure in space has **plane symmetry** if there is a symmetry plane X such that reflection in the plane maps the figure onto itself. (See Exercise 17, page 580.) Most living creatures have a single plane of symmetry. Such symmetry is called *bilateral symmetry*. The photographs on the next page illustrate bilateral symmetry.

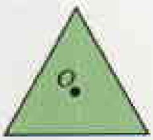
Some geometric solids have more than one symmetry plane. For example, the regular hexagonal prism shown has seven symmetry planes, two of which are shown. It also has six-fold rotational symmetry because rotating it 60°, 120°, 180°, 240°, 300°, or 360° about the line k (called the *axis of symmetry*) maps the prism onto itself.



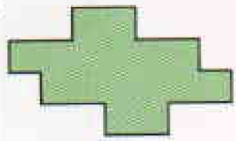
Classroom Exercises

Tell how many symmetry lines each figure has. In Exercise 2, O is the center of the equilateral triangle.

1. 

2. 

3. 

4. 

5. Which figures above have point symmetry?

6. Describe all of the rotational symmetries of the figure in Exercise 2.

7. Describe all of the rotational symmetries of the figure in Exercise 3.

Draw each figure on the chalkboard and describe all of its symmetries.

8. isosceles triangle

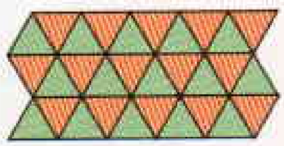
9. parallelogram

10. rectangle

11. rhombus

12. Imagine that the pattern shown fills the entire plane. Does the pattern have the symmetry named?

a. translational symmetry
b. line symmetry
c. point symmetry
d. rotational symmetry

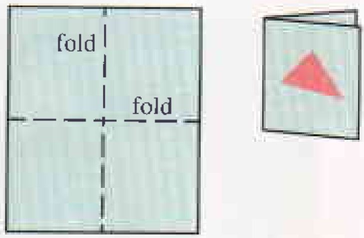


13. How many planes of symmetry does the given solid have?

a. a rectangular solid
b. a sphere
c. a regular square pyramid

14. Where are the centers of the rotational symmetries for the tessellations in the middle of page 610?

15. Fold a piece of paper into quarters as shown. Cut out a scalene triangle that does not touch any of the edges. Unfold the paper. Describe the symmetries of the design.

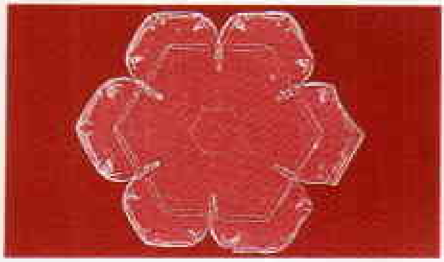


Written Exercises

Consider the object shown in each photograph as a plane figure.
a. State how many symmetry lines each figure has.
b. State whether or not the figure has a symmetry point.
c. List all the rotational symmetries of each figure between 0° and 360°.

A 1.

2.



3.

4.

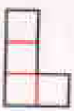
5. Which capital letters of the alphabet have just one line of symmetry? (One answer is "D".)

6. Which capital letters of the alphabet have two lines of symmetry?

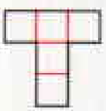
7. Which capital letters of the alphabet have a point of symmetry?

Make a tessellation of the given figure.

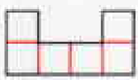
8.



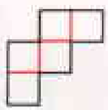
9.



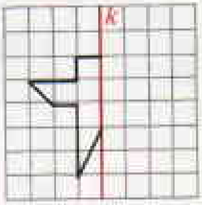
10.



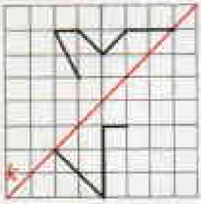
11.



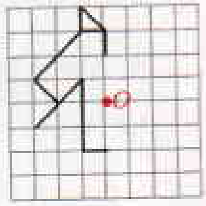
Copy the figure shown. Then complete the figure so that it has the specified symmetries.

12. 

symmetry in line k

13. 

symmetry in line k

14. 

symmetry in point O

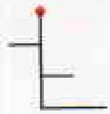
Copy the figure shown. Then complete the figure so that it has the specified symmetries.

B **15.** 

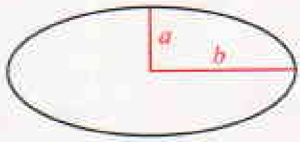
60°, 120°, and 180° rotational symmetry

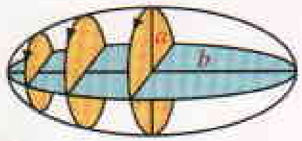
16. 

90°, 180°, and 270° rotational symmetry

17. 

2 symmetry lines and 1 symmetry point





18. a. An octopus has one symmetry. Describe it.

b. If you disregard the eyes and mouth of an octopus, it has many symmetries. Describe them.

19. a. Describe the symmetries of the ellipse shown.

b. If the ellipse is rotated in space about one of its symmetry lines, an ellipsoid (an egg-like figure) is formed. Its volume is $V = \frac{4}{3}\pi a^2 b$. Interpret this formula when $a = b$.

c. Describe the symmetries of an ellipsoid.

20. Tell whether a tessellation can be made with the given figure.

a. A regular hexagon

b. A scalene triangle

c. A regular pentagon

d. A nonisosceles trapezoid

In Exercises 21–23 draw the figure if there is one that meets the conditions. Otherwise write *not possible*.

21. A trapezoid with **(a)** no symmetry, **(b)** one symmetry line, **(c)** a symmetry point.

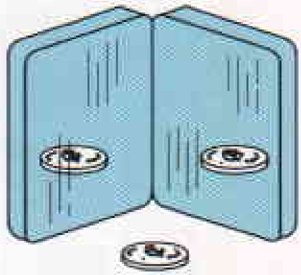
22. A parallelogram with **(a)** four symmetry lines, **(b)** just two symmetry lines, **(c)** just one symmetry line.

23. An octagon with **(a)** eight rotational symmetries, **(b)** just four rotational symmetries, **(c)** only point symmetry.

24. If you use tape to hinge together two pocket mirrors as shown and place the mirrors at a 120° angle, then a coin placed between the mirrors will be reflected, giving a pattern with 120° and 240° rotational symmetry.

a. What kinds of symmetries occur when the mirrors are at a right angle?

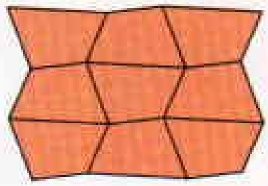
b. Experiment by forming various angles with two mirrors. Be sure to try 60°, 45°, and 30° angles. Record the number of coins you see, including the actual coin.



25. You can make a tessellation by tracing around *any* quadrilateral, placing copies of the quadrilateral systematically as shown.

a. The tessellation shown has many symmetry points but none of these are at vertices of the quadrilateral. Where are they?

b. What other kind of symmetry does this mosaic have?



26. A figure has 60° rotational symmetry. What other rotational symmetries *must* it have? Explain your answer.

C 27. Show that if a hexagon has point symmetry, then its opposite sides must be parallel.

28. A figure has 50° rotational symmetry. What other rotational symmetries *must* it have? Explain your answer.

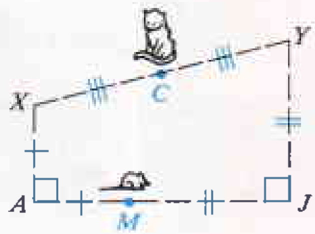
★ 29. Tell how many planes of symmetry and axes of rotation each solid has.

a. a right circular cone **b.** a cube

c. a regular tetrahedron (a pyramid formed by four equilateral triangles)

Challenges

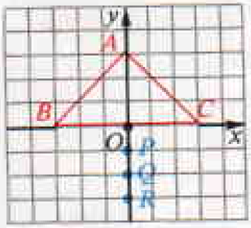
1. A mouse moves along $\overline{AJ}$. For any position M of the mouse, X and Y are such that $\overline{AX} \perp \overline{AJ}$ with $AX = AM$, and $\overline{JY} \perp \overline{AJ}$ with $JY = JM$. The cat is at C, the midpoint of $\overline{XY}$. Describe the locus of the cat as the mouse moves from A to J.



2. Points O, A, B, and C lie on a number line with coordinates 0, 8, 12, and 26. Take any point P not on the line. Draw $\overline{PA}$ and label its midpoint Q. Draw $\overline{QB}$ and label its midpoint R. Draw $\overline{PC}$ and label its midpoint S. Draw $\overrightarrow{SR}$. What is the coordinate of the point where $\overrightarrow{SR}$ intersects the number line?

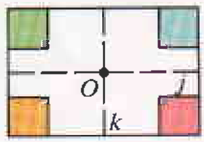
Self-Test 2

For Exercises 1–6, refer to the figure.

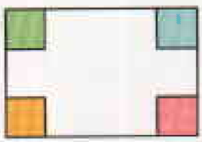


1. $R_x \circ \mathcal{R}_{O,\,90}: B \rightarrow$ ___?___
2. $R_x \circ H_O: A \rightarrow$ ___?___
3. $\mathcal{R}_{O,\,110} \circ \mathcal{R}_{O,\,70}: C \rightarrow$ ___?___
4. $D_{O,\,\frac{1}{2}} \circ D_{R,\,\frac{1}{2}}: P \rightarrow$ ___?___
5. What is the symmetry line of $\triangle ABC$?
6. Does $\triangle ABC$ have point symmetry?
7. For any transformation T, $T^{-1} \circ T: P \rightarrow$ ___?___.
8. The composite of any transformation T and the identity is ___?___.
9. If line a is parallel to line b, then the composite $R_a \circ R_b$ is a ___?___.
10. Give the inverse of each transformation.
    a. $D_{O,\,5}$ **b.** $\mathcal{R}_{O,\,-70}$ **c.** R_y **d.** $S:(x,\ y) \rightarrow (x + 2,\ y - 3)$
11. How many lines of symmetry does a regular hexagon have?

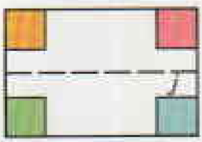
Extra Symmetry Groups



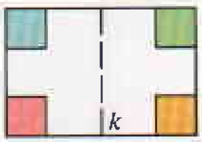
Cut out a cardboard or paper rectangle and color each corner with a color of its own on both front and back. Also on the front and back draw symmetry lines j and k and label symmetry point O. The rectangle has four symmetries: I, R_j, R_k, and H_O. The effect of each of these on the original rectangle is shown below.



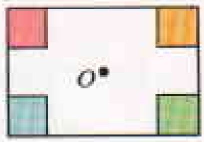
Effect of I:
Rectangle unchanged



Effect of R_j

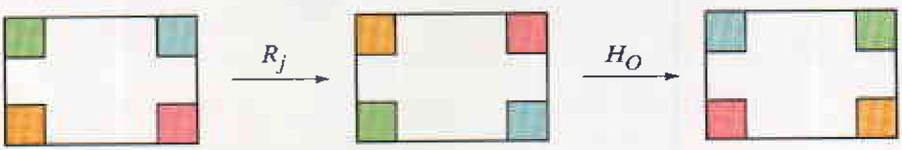


Effect of R_k



Effect of H_O

Our goal is to see how the four symmetries of the rectangle combine with each other. For example, if the original rectangle is mapped first by R_j and then by H_O, the images look like this:



Mapping the rectangle by R_j and then by H_O has the same effect as the single symmetry R_k, so $H_O \circ R_j = R_k$. We can record this fact in a table resembling a multiplication table. Follow the row for R_j to where it meets the column for H_O, and enter the product $H_O \circ R_j$, which is R_k.

$\circ$	I	R_j	R_k	H_O
I				
R_j				R_k
R_k				
H_O				

We can determine other products of symmetries in the same way, but sometimes short cuts can be used. For example, we know that

(1) $R_j \circ R_j = I$ and $R_k \circ R_k = I$ (Why?)
(2) $H_O \circ H_O = I$ (Why?)
(3) $R_j \circ R_k = H_O$ and $R_k \circ R_j = H_O$
(Corollary to Theorem 14-8)

Also we know that the product of any symmetry and the identity is that same symmetry. The completed table is shown at the right.

$\circ$	I	R_j	R_k	H_O
I	I	R_j	R_k	H_O
R_j	R_j	I	H_O	R_k
R_k	R_k	H_O	I	R_j
H_O	H_O	R_k	R_j	I

By studying the table you can see that the symmetries of the rectangle have these four properties, similar to the properties of nonzero real numbers under multiplication:

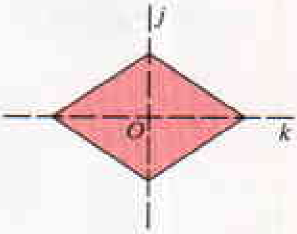
(1) The product of two symmetries is another symmetry.
(2) The set of symmetries contains the identity.
(3) Each symmetry has an inverse that is also a symmetry. (In this example each symmetry is its own inverse.)
(4) Forming products of transformations is an associative operation: $A \circ (B \circ C) = (A \circ B) \circ C$ for any three symmetries A, B, and C.

A set of symmetries with these four properties is called a symmetry *group*. Symmetry groups are used in crystallography, and more general groups are important in physics and advanced mathematics. The exercises that follow illustrate the fact that the symmetries of any figure form a group.

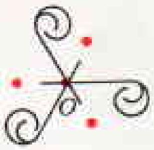
Exercises

1. An isosceles triangle has just two symmetries, including the identity. Make a 2 by 2 group table showing how these symmetries combine.

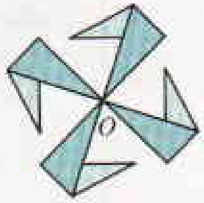
2. a. List the four symmetries of the rhombus shown. (Include the identity.)
b. Make a group table showing all products of two symmetries.
c. Is your table in part (b) identical to the table of symmetries for the rectangle?



3. Make a group table for the three symmetries of this figure.



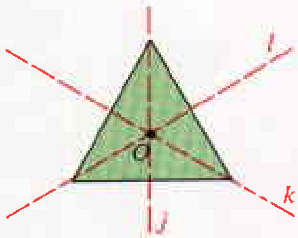
4. Make a group table for the four symmetries of this figure.



5. A transformation that is its own inverse is called a *self-inverse*.
   a. How many of the four symmetries of the figure in Exercise 4 are self-inverses?
   b. How many of the four symmetries of the rectangle are self-inverses?

6. A symmetry group is called *commutative* if $A \circ B = B \circ A$ for every pair of symmetries A and B in the group. The symmetry group of the rectangle is commutative, as you can see from the completed table. (For example, $H_O \circ R_j$ and $R_j \circ H_O$ are both equal to R_k.) Tell whether the groups in Exercises 3 and 4 are commutative or not.

7. An equilateral triangle has three rotational symmetries (I, $\mathcal{R}_{O,120}$, and $\mathcal{R}_{O,240}$) and three line symmetries (R_j, R_k, and R_l).
   a. Make a group table for these six symmetries.
   b. Give an example which shows that this group is *not* commutative.



8. A square has four rotational symmetries (including the identity) and four line symmetries. Make a group table for these eight symmetries. Is this a commutative group?

9. The four rotational symmetries of the square satisfy the four requirements for a group, and so they are called a *subgroup* of the full symmetry group. (Notice that the identity is one of these rotational symmetries and that the product of two rotations is another rotation in the subgroup.)
   a. Do the four line symmetries of the square form a subgroup?
   b. Does the symmetry group of the equilateral triangle have a subgroup?
   c. Which two symmetries of the figure in Exercise 4 form a subgroup?

10. The tessellation with fish on page 610 has translational symmetry. Let S be the horizontal translation mapping each fish to the fish of the same color to its right, and let T be the vertical translation mapping each fish to the fish of the same color directly above.
   a. Describe the mapping S^3. Is it a symmetry of the pattern?
   b. Describe T^{-1}. Is it a symmetry?
   c. Describe $S \circ T$. Is it a symmetry?
   d. How many symmetries does the tessellation have?
   e. Does this set of symmetries satisfy the four requirements for a group?

Chapter Summary

1. A transformation is a one-to-one mapping from the whole plane to the whole plane. If the transformation S maps P to P', we write $S:P \rightarrow P'$ or $S(P) = P'$.
2. The word "mapping" is used in geometry as the word "function" is used in algebra. If the function f maps every number to its square we write $f:x \rightarrow x^2$ or $f(x) = x^2$.
3. An isometry is a transformation that preserves distance. An isometry maps any figure to a congruent figure.
4. Some basic isometries are:

   Reflection in a line. R_j is a reflection in line j.

   Translation or glide. $T:(x, y) \rightarrow (x + a, y + b)$ is a translation.

   Rotation about a point. $\mathcal{R}_{O,x}$ is a rotation counterclockwise about O through $x°$. H_O is a half-turn about O.

   Glide reflection. A glide followed by a reflection in a line parallel to the glide yields a glide reflection.
5. A dilation maps any figure to a similar figure. $D_{O,k}$ is a dilation with center O and nonzero scale factor k. A dilation is an isometry if $|k| = 1$.
6. Properties of figures that are preserved by a transformation are said to be invariant under that transformation. Invariant properties are checked in the table below.

	Distance	Angle Measure	Parallelism	Ratio of distances	Area
Isometry:	✓	✓	✓	✓	✓
Dilation:		✓	✓	✓	

7. The combination of one mapping followed by another is called a composite or product of mappings. The mapping A followed by B is written $B \circ A$.
8. A composite of isometries is an isometry.
   A composite of reflections in two parallel lines is a translation.
   A composite of reflections in two intersecting lines is a rotation.
   A composite of reflections in two perpendicular lines is a half-turn.
9. The identity transformation I keeps all points fixed. A transformation S followed by its inverse S^{-1} is equal to the identity.
10. A symmetry of a figure is an isometry that maps the figure onto itself. Figures can have line symmetry, point symmetry, and rotational symmetry. A tessellation, or covering of the plane with congruent figures, may also have translational and glide reflection symmetry. Solid figures in space can have planes of symmetry and rotational symmetry about an axis.

Chapter Review

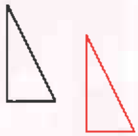
1. If isometry S maps A to A' and B to B', then $\overline{AB}$ _?_ $\overline{A'B'}$. **14–1**
2. If $f(x) = 3x$, find the image and preimage of 6.
3. **a.** If $S:(x, y) \to (2x, y - 2)$, find the image and preimage of (3, 3).
   b. Is S an isometry?
4. Find the image of $(-7, 5)$ when reflected in **(a)** the x-axis, **(b)** the y-axis, and **(c)** the line $y = x$. **14–2**
5. Draw the line $y = 2x + 1$ and its image under reflection in the y-axis.
6. **a.** If translation $T:(5, 5) \to (7, 1)$, then $T:(x, y) \to$(_?_, _?_) **14–3**
   b. Is distance invariant under T?
   c. Is angle measure invariant under T?
   d. Is area invariant under T?
7. Find the image of $(7, -2)$ under the glide reflection that moves all points 5 units to the right and then reflects all points in the x-axis.
8. Plot the points $A(3, 2)$, $B(-1, 1)$, and $C(1, -3)$. Label the origin O. Draw $\triangle ABC$ and its images under **(a)** $\mathcal{R}_{O, 90}$ and **(b)** H_O. **14–4**
9. Which of the given rotations are equal to $\mathcal{R}_{O, 140}$?
   a. $\mathcal{R}_{O, 500}$ **b.** $\mathcal{R}_{O, -140}$ **c.** $\mathcal{R}_{O, -220}$
10. If O is the origin then the dilation $D_{O, 2}:(3, -2) \to$ (_?_, _?_). **14–5**
11. Find the image of (3, 1) under a dilation with center (0, 4) and scale factor $\frac{1}{3}$.
12. Find the image of (3, 1) under the following transformations: **14–6**
    a. $R_x \circ R_y$ **b.** $R_y \circ H_O$ **c.** $R_x \circ \mathcal{R}_{O, -90}$

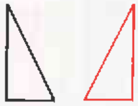
Complete.

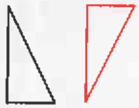
13. If $T:(x, y) \to (x - 1, y + 6)$, then $T^{-1}:(x, y) \to$ (_?_, _?_). **14–7**
14. The inverse of $D_{O, 4}$ is $D_{?,?}$.
15. $R_j \circ R_j =$ _?_
16. $\mathcal{R}_{O, 75} \circ \mathcal{R}_{O, ?} = I$
17. Does a scalene triangle have line symmetry? **14–8**
18. Does a rectangle have point symmetry?
19. Does a regular octagon have 90° rotational symmetry?
20. Name a figure that has 72° rotational symmetry.

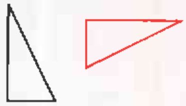
Chapter Test

State whether the transformation mapping the black triangle to the red triangle is a reflection, a translation, a glide reflection, or a rotation.

1. 

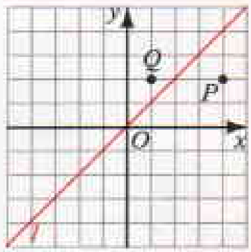
2. 

3. 

4. 

5. If $f(x) = \frac{1}{2}x + 3$, find the image and preimage of 4.

Give the coordinates of the image of point P under the transformation specified.



6. R_l

7. $\mathcal{R}_{O,\,-90}$

8. $D_{O,\,\frac{1}{2}}$

9. $H_O \circ R_x$

10. $\mathcal{R}_{O,\,90} \circ \mathcal{R}_{O,\,90}$

11. $D_{Q,\,\frac{1}{3}}$

12. $R_l \circ D_{Q,\,-2}$

13. $R_l \circ (R_y \circ R_x)$

Give the inverse of each transformation.

14. H_O

15. R_x

16. $D_{O,\,-2}$

T is the translation mapping (4, 1) to (6, 2). Find the coordinates of the image of the origin under each mapping.

17. T

18. T^3

19. T^{-1}

Classify each statement as true or false.

20. All regular polygons have rotational symmetry.

21. 180° rotational symmetry is the same as point symmetry.

22. All regular n-gons have exactly n symmetry lines.

23. A figure that has two intersecting lines of symmetry must have rotational symmetry.

24. **a.** Is a half-turn a transformation? Is it an isometry?
b. Name three properties that are invariant under a half-turn.

25. A line has slope 2. What is the slope of the image of the line under a:
a. reflection in the x-axis?
b. reflection in the line $y = x$?
c. dilation $D_{O,\,3}$?

Preparing for College Entrance Exams

Strategy for Success

Try to work quickly and accurately on exam questions. Do not take time to double-check your answers unless you finish all the questions before the deadline. Skip questions that are too difficult for you, and spend no more than a few minutes on each question.

Indicate the best answer by writing the appropriate letter.

1. Find an equation of the perpendicular bisector of the segment joining $(3, -1)$ and $(-1, 7)$.
(A) $x + 2y = 7$ **(B)** $x - 2y = -5$ **(C)** $2x + y = -5$
(D) $2x + y = 5$ **(E)** $2x - y = -1$

2. A circle has a diameter with endpoints $(0, -8)$ and $(-6, -16)$. An equation of the circle is:
(A) $(x + 3)^2 + (y + 12)^2 = 25$ **(B)** $(x + 3)^2 + (y + 12)^2 = 100$
(C) $(x - 3)^2 + (y - 12)^2 = 25$ **(D)** $(x - 3)^2 + (y - 12)^2 = 100$
(E) $(x + 6)^2 + (y + 24)^2 = 100$

3. The point $(\frac{1}{2}, -\frac{1}{2})$ lies on line t. Which of the following allow you to find an equation for t?
I. slope of t is -3 II. x-intercept of t is 7 III. t is parallel to $4x - 5y = 7$
(A) I only **(B)** III only **(C)** I and III only **(D)** II only **(E)** I, II, and III

4. Given $A(-3, 5)$, $B(0, -4)$, $C(2, 5)$, and $D(-6, -1)$, find the intersection point of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.
(A) $(6, 23)$ **(B)** $(2, -10)$ **(C)** $(-2, 2)$ **(D)** $(-18, 5)$ **(E)** cannot be determined

5. What is the best name for quadrilateral $WXYZ$ with vertices $W(-3, -2)$, $X(-5, 2)$, $Y(1, 5)$, and $Z(3, 1)$?
(A) isosceles trapezoid **(B)** parallelogram **(C)** rectangle
(D) rhombus **(E)** square

6. Two vertices of an isosceles right triangle are $(0, 0)$ and $(j, 0)$. The third vertex cannot be:
(A) $(0, j)$ **(B)** $(0, -j)$ **(C)** (j, j) **(D)** $\left(\frac{j}{2}, \frac{j}{2}\right)$ **(E)** $\left(\frac{j}{2}, j\right)$

7. What is the image of $(-2, 3)$ under reflection in the line $y = x$?
(A) $(3, -2)$ **(B)** $(2, 3)$ **(C)** $(-2, -3)$ **(D)** $(2, -3)$ **(E)** $(-3, 2)$

8. Find the preimage of $(0, 0)$ under $D_{P, \frac{1}{4}}$, where P is the point $(-1, 1)$.
(A) $(-4, 4)$ **(B)** $(-\frac{3}{4}, \frac{3}{4})$ **(C)** $(-\frac{1}{4}, \frac{1}{4})$ **(D)** $(4, -4)$ **(E)** $(3, -3)$

9. A regular pentagon does *not* have:
(A) line symmetry **(B)** point symmetry **(C)** 360° rotational symmetry
(D) 216° rotational symmetry **(E)** 72° rotational symmetry

10. If $CDEF$ is a square with vertices labeled counterclockwise, then $\mathcal{R}_{C, -450}: \overline{CF} \rightarrow$ __?__.
(A) $\overline{FE}$ **(B)** $\overline{ED}$ **(C)** $\overline{CF}$ **(D)** $\overline{CD}$ **(E)** none of these

Cumulative Review: Chapters 1-14

True-False Exercises

Classify each statement as true or false.

A

1. Three given points are always coplanar.
2. Each interior angle of a regular n-gon has measure $\frac{(n-2)180}{n}$.
3. If $\triangle RST \cong \triangle RSV$, then $\angle SRT \cong \angle SRV$.
4. The contrapositive of a true conditional is sometimes false.
5. Corresponding parts of similar triangles must be congruent.
6. An acute angle inscribed in a circle must intercept a minor arc.
7. In a plane the locus of points equidistant from M and N is the midpoint of $\overline{MN}$.
8. If a cylinder and a right prism have equal base areas and equal heights, then they have equal volumes.
9. A triangle with vertices $(a, 0)$, $(-a, 0)$, and $(0, a)$ is equilateral.
10. If the slopes of two lines have opposite signs, the lines are perpendicular.
11. $R_k \circ R_k = I$
12. If a figure has 90° rotational symmetry, then it also has point symmetry.

B

13. A point lies on the bisector of $\angle ABC$ if and only if it is equidistant from A and C.
14. In $\triangle RST$, if $RS < ST$, then $\angle R$ must be the largest angle of the triangle.
15. A triangle with sides of length $2x$, $3x$, and $4x$ must be obtuse.
16. In a right triangle, the altitude to the hypotenuse is always the shortest of the three altitudes.
17. Given a segment of length t, it is possible to construct a segment of length $t\sqrt{3}$.
18. If an equilateral triangle and a regular hexagon are inscribed in a circle, then the ratio of their areas is 1:2.
19. The lateral area of a cone can be equal to the area of the base of the cone.
20. The circle $(x + 3)^2 + (y - 2)^2 = 4$ is tangent to the line $x = -1$.

Multiple-Choice Exercises

Write the letter that indicates the best answer.

A

1. The measure of an interior angle of a regular decagon is:
   a. 36 **b.** 108 **c.** 72 **d.** 144
2. Which of the following is *not* a method for proving two triangles congruent?
   a. HL **b.** AAS **c.** SSA **d.** SAS

3. The median to the hypotenuse of a right triangle divides the triangle into two triangles that are both:
   a. similar b. isosceles c. scalene d. right

4. Which proportion is *not* equivalent to $\frac{a}{b} = \frac{c}{d}$?
   a. $\frac{a}{c} = \frac{b}{d}$ b. $\frac{b}{a+b} = \frac{d}{c+d}$ c. $\frac{b}{a} = \frac{d}{c}$ d. $\frac{a}{d} = \frac{c}{b}$

5. For every acute angle X:
   a. $\cos X < \sin X$ b. $\cos X > \tan X$ c. $\tan X > 1$ d. $\cos X < 1$

B 6. If A, B, and C are points on $\odot O$, $\overline{AC}$ is a diameter, and $m\angle AOB = 60$, then $m\angle ACB =$
   a. 30 b. 60 c. 90 d. 120

7. A rectangle with perimeter 30 and area 44 has length:
   a. $2\sqrt{11}$ b. 8 c. 11 d. 10

8. A regular hexagon with perimeter 24 has area:
   a. $24\sqrt{3}$ b. $16\sqrt{3}$ c. $48\sqrt{3}$ d. $32\sqrt{3}$

9. In $\odot O$, $m\widehat{AB} = 90$ and $OA = 6$. The region bounded by $\overline{AB}$ and $\widehat{AB}$ has area:
   a. $3\pi - 6$ b. $9\pi - 36$ c. $9\pi - 18$ d. $36\pi - 6\sqrt{2}$

10. Two regular octagons have sides of length $6\sqrt{3}$ and 9. The ratio of their areas is:
   a. $2\sqrt{3}:3$ b. $4:3$ c. $2:3$ d. $8\sqrt{3}:9$

11. If F is the point $(-3, 5)$ and G is the point $(0, -4)$, then an equation of $\overleftrightarrow{FG}$ is:
   a. $y = -\frac{1}{3}x + 4$ b. $y = -3x - 4$ c. $y = \frac{1}{3}x + 4$ d. $y = -3x + 4$

Completion Exercises

Write the correct word, number, phrase, or expression.

A 1. If $5x - 1 = 14$, then the statement $5x = 15$ is justified by the __?__.

2. If two parallel lines are cut by a transversal, then __?__ angles are congruent, __?__ angles are congruent, and __?__ angles are supplementary.

3. The measures of two angles of a triangle are 56 and 62. The measure of the largest exterior angle of the triangle is __?__.

4. In $\triangle BEV$ with $m\angle B = 53$ and $m\angle E = 63$, the longest side is __?__.

5. The area of a triangle with vertices $(-2, 0)$, $(9, 0)$, and $(3, 6)$ is __?__.

6. The distance between $(-5, -2)$ and $(1, -6)$ is __?__.

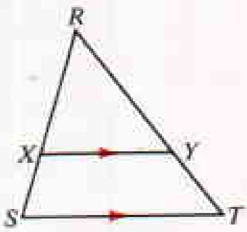
7. If $j \perp k$ and line j has slope $\frac{2}{3}$, then k has slope __?__.

8. If A is $(-8, 3)$ and B is $(-4, -1)$, then the midpoint of $\overline{AB}$ is (__?__, __?__).

B 9. If O is the origin, then $R_x \circ \mathcal{R}_{O, 90}: (-2, 5) \rightarrow$ (__?__, __?__).

10. If $RX = 18$, $XS = 10$, and $RT = 35$, then $YT =$ __?__.

11. If $RX = 16$, $XS = 8$, and $XY = 15$, then $ST =$ __?__.



Exs. 10, 11

12. If a dart thrown at a 64-square checkerboard lands on the board, the probability that it lands on a black square is _?_. The probability that it lands on one of the four central squares is _?_.

13. In $\triangle ABC$, $\overline{AB} \perp \overline{BC}$, $AB = 15$, and $BC = 8$. Then the exact value of $\sin C$ is _?_.

14. A tree 5 m tall casts a shadow 8 m long. To the nearest degree, the angle of elevation of the sun is _?_. (Use the table on page 311.)

15. A trapezoid with sides 8, 8, 8, and 10 has area _?_.

16. A circle with area 100π has circumference _?_.

17. A cone with radius 9 and slant height 15 has volume _?_ and lateral area _?_.

18. A sphere with surface area 144π cm^2 has volume _?_.

19. If $B = (2, 0)$, then $D_{B,-2}:(1, 1) \rightarrow$ (_?_, _?_).

C 20. If each edge of a regular triangular pyramid is 6 cm, then the pyramid has total area _?_ and volume _?_.

21. A plane parallel to the base of a cone and bisecting the altitude divides the cone into two parts whose volumes have the ratio _?_.

Always-Sometimes-Never Exercises

Write A, S, or N to indicate your answer.

A 1. Vertical angles are _?_ adjacent angles.

2. If J is a point outside $\odot P$ and $\overline{JA}$ and $\overline{JB}$ are tangents to $\odot P$ with A and B on $\odot P$, then $\triangle JAB$ is _?_ scalene.

3. A conclusion based on inductive reasoning is _?_ correct.

4. Two right triangles with congruent hypotenuses are _?_ congruent.

5. If the diagonals of a quadrilateral are perpendicular bisectors of each other, then the quadrilateral is _?_ a rhombus.

6. If $\triangle RST$ is a right triangle with hypotenuse $\overline{RS}$, then $\sin R$ and $\cos S$ are _?_ equal.

7. A circle _?_ contains three collinear points.

8. A lateral edge of a regular pyramid is _?_ longer than the slant height.

9. Transformations are _?_ isometries.

10. Under a half-turn about point O, point O is _?_ mapped onto itself.

B 11. If the measures of three consecutive angles of a quadrilateral are 58, 122, and 58, then the diagonals _?_ bisect each other.

12. A triangle with sides of length x, $x + 2$, and $x + 4$ is _?_ an acute triangle.

13. If $\overset{\frown}{RS}$ and $\overset{\frown}{XY}$ are arcs of $\odot O$ and $m\overset{\frown}{RS} < m\overset{\frown}{XY}$, then RS and XY are _?_ equal.

14. The center of the circle that can be circumscribed about a given triangle is _?_ outside the triangle.
15. Given two segments with lengths r and s, it is _?_ possible to construct a segment of length $\frac{3}{4}\sqrt{2rs}$.
16. A median of a triangle _?_ separates the triangle into two triangles with equal areas.
17. A composite of reflections in two lines is _?_ a translation.

Construction Exercises

A 1. Construct an angle of measure $22\frac{1}{2}$.

2. Draw a circle O and choose a point T on $\odot O$. Construct the tangent to $\odot O$ at T.
3. Draw a large triangle. Inscribe a circle in the triangle.

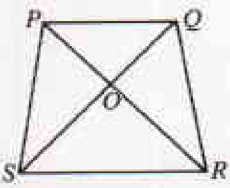
For Exercises 4–7, draw two long segments. Let their lengths be x and y, with $x > y$.

4. Construct a segment of length $\frac{1}{2}(x + y)$.

B 5. Construct a rectangle with width y and diagonal x.

6. Construct any triangle with area xy.
7. Construct a segment with length $\sqrt{3xy}$.
8. Draw a very long $\overline{AB}$. Construct a rectangle with perimeter AB and sides in the ratio 3:2.

Proof Exercises



Exs. 1–3

A 1. Given: $\overline{PQ} \parallel \overline{RS}$
Prove: $\frac{PO}{RO} = \frac{PQ}{RS}$

2. Given: $\overline{PR} \perp \overline{QS}$; $\overline{PS} \cong \overline{QR}$; $\overline{OS} \cong \overline{OR}$
Prove: $\angle PSO \cong \angle QRO$

B 3. Given: $\angle OSR \cong \angle ORS$; $\angle OPQ \cong \angle OQP$
Prove: $\triangle PSR \cong \triangle QRS$

4. Prove: The diagonals of a rectangle intersect to form four congruent segments.
5. Use coordinate geometry to prove that the triangle formed by joining the midpoints of the sides of an isosceles triangle is an isosceles triangle.

C 6. Use an indirect proof to show that a trapezoid cannot have two pairs of congruent sides.

7. Prove: If two coplanar circles intersect in two points, then the line joining those points bisects a common tangent segment.