5 QUADRILATERALS

All artists seek ways to express their own visions of life and the world. Here, Hans Hofmann has created a painting composed entirely of quadrilaterals.

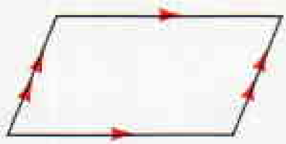
Parallelograms

Objectives

1. Apply the definition of a parallelogram and the theorems about properties of a parallelogram.
2. Prove that certain quadrilaterals are parallelograms.
3. Apply theorems about parallel lines and the segment that joins the midpoints of two sides of a triangle.

5-1 *Properties of Parallelograms*

A **parallelogram** ($\square$) is a quadrilateral with both pairs of opposite sides parallel. The following theorems state some properties common to all parallelograms. Your proofs of these theorems (Written Exercises 13–15) will be based on what you have learned about parallel lines and congruent triangles.



Theorem 5-1

Opposite sides of a parallelogram are congruent.

Given: $\square EFGH$

Prove: $\overline{EF} \cong \overline{HG}$; $\overline{FG} \cong \overline{EH}$

Plan for Proof: Draw $\overline{EG}$ to form triangles with corresponding sides $\overline{EF}$ and $\overline{HG}$, $\overline{FG}$ and $\overline{EH}$. Use the pairs of alternate interior angles $\angle 1$ and $\angle 2$, $\angle 3$ and $\angle 4$, to prove the triangles congruent by ASA.

Theorem 5-2

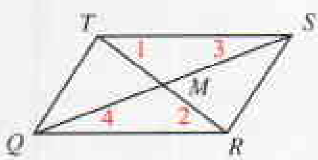
Opposite angles of a parallelogram are congruent.

Theorem 5-3

Diagonals of a parallelogram bisect each other.

Given: $\square QRST$ with diagonals $\overline{QS}$ and $\overline{TR}$

Prove: $\overline{QS}$ and $\overline{TR}$ bisect each other.

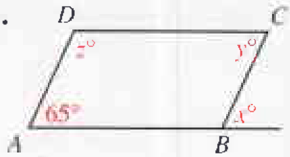


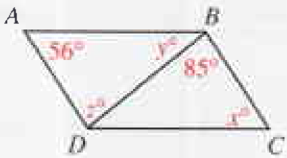
Plan for Proof: You can prove that $\overline{QM} \cong \overline{MS}$ and $\overline{RM} \cong \overline{MT}$ by showing that they are corresponding parts of congruent triangles. Since $\overline{QR} \cong \overline{TS}$ by Theorem 5-1, you can show that $\triangle QMR \cong \triangle SMT$ by ASA.

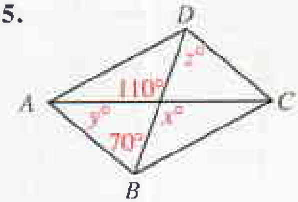
Classroom Exercises

1. Quad. *GRAM* is a parallelogram.
   a. Why is $\angle G$ supplementary to $\angle M$?
   b. Why is $\angle M$ supplementary to $\angle A$?
   c. Complete: Consecutive angles of a parallelogram are __?__, while opposite angles are __?__.
2. Suppose that $\angle M$ is a right angle. What can you deduce about angles G, R, and A?

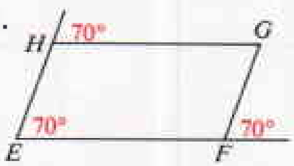
In Exercises 3–5 quad. *ABCD* is a parallelogram. Find the values of *x*, *y*, and *z*.

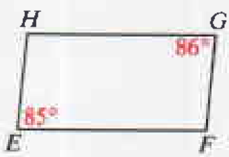
3. 

4. 

5. 

Must quad. *EFGH* be a parallelogram? Can it be a parallelogram? Explain.

6. 

7. 

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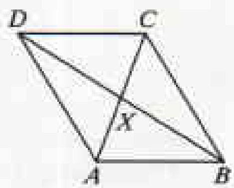
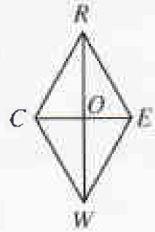


Quad. *ABCD* is a parallelogram. Name the principal theorem or definition that justifies the statement.

9. $\overline{AD} \parallel \overline{BC}$
10. $\angle ADX \cong \angle CBX$
11. $m\angle ABC = m\angle CDA$
12. $\overline{AD} \cong \overline{BC}$
13. $AX = \frac{1}{2}AC$
14. $DX = BX$
15. Draw a quadrilateral that isn't a parallelogram but does have two 60° angles opposite each other.
16. State each theorem in if-then form. (Begin "If a quadrilateral is a ....")
    a. Theorem 5-1 b. Theorem 5-2 c. Theorem 5-3
17. a. Draw any two segments, $\overline{AC}$ and $\overline{BD}$, that bisect each other at O. What appears to be true of quad. *ABCD*?
    b. This exercise investigates the converse of what theorem?
18. Draw two segments that are both parallel and congruent. Connect their endpoints to form a quadrilateral. What appears to be true of the quadrilateral?

Written Exercises

Exercises 1–4 refer to ▱*CREW*.

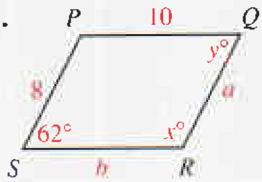


A

1. If $OE = 4$ and $WE = 8$, name two segments congruent to $\overline{WE}$.
2. If $\overline{WR} \perp \overline{CE}$, name all angles congruent to $\angle RCE$.
3. If $\overline{WR} \perp \overline{CE}$, name all segments congruent to $\overline{WE}$.
4. If $RE = EW$, name all angles congruent to $\angle ERW$.

In Exercises 5–10 quad. *PQRS* is a parallelogram. Find the values of *a*, *b*, *x*, and *y*.

5.



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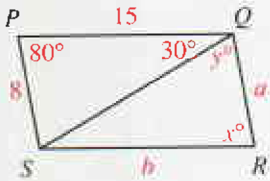

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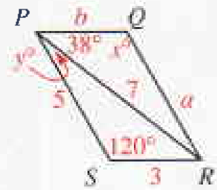

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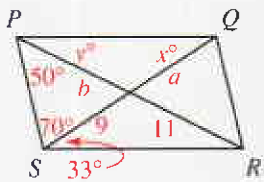

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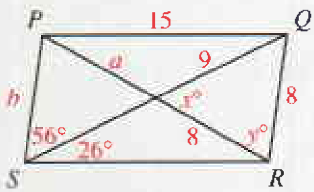

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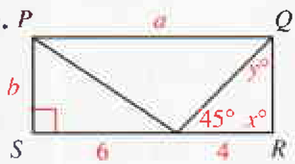
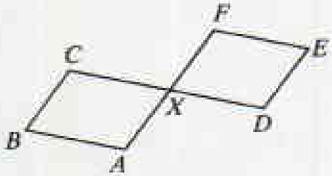


11. Find the perimeter of ▱*RISK* if $RI = 17$ and $IS = 13$.
12. The perimeter of ▱*STOP* is 54 cm, and $\overline{ST}$ is 1 cm longer than $\overline{SP}$. Find *ST* and *SP*.
13. Prove Theorem 5-1.
14. Prove Theorem 5-2. (Draw and label a diagram. List what is given and what is to be proved.)
15. Prove Theorem 5-3.

16. Given: *ABCX* is a ▱;
DXFE is a ▱.
Prove: $\angle B \cong \angle E$



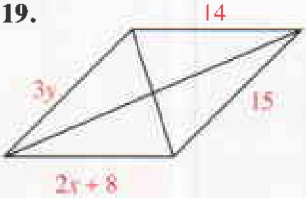
The coordinates of three vertices of $\square ABCD$ are given. Plot the points and find the coordinates of the fourth vertex.

17. $A(1, 0)$, $B(5, 0)$, $C(7, 2)$, $D(\underline{\ ?\ }, \underline{\ ?\ })$

18. $A(3, 2)$, $B(8, 2)$, $C(\underline{\ ?\ }, \underline{\ ?\ })$, $D(0, 5)$

Each figure in Exercises 19–24 is a parallelogram with its diagonals drawn. Find the values of x and y.

19.



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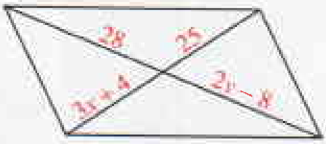

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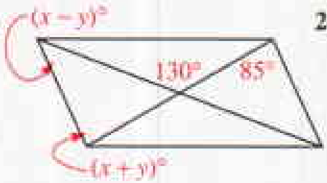
B **22.**



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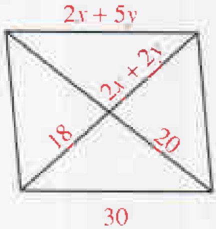

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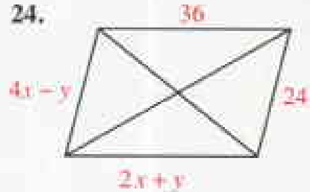
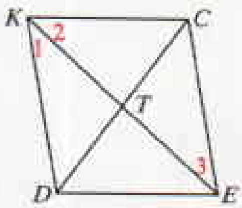


Quad. *DECK* is a parallelogram. Complete.

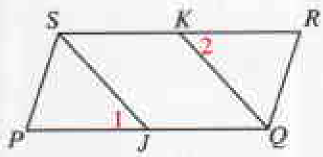


25. If $KT = 2x + y$, $DT = x + 2y$, $TE = 12$, and $TC = 9$, then $x = \underline{\ ?\ }$ and $y = \underline{\ ?\ }$.

26. If $DE = x + y$, $EC = 12$, $CK = 2x - y$, and $KD = 3x - 2y$, then $x = \underline{\ ?\ }$, $y = \underline{\ ?\ }$, and the perimeter of $\square DECK = \underline{\ ?\ }$.

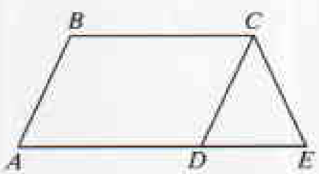
27. If $m\angle 1 = 3x$, $m\angle 2 = 4x$, and $m\angle 3 = x^2 - 70$, then $x = \underline{\ ?\ }$ and $m\angle CED = \underline{\ ?\ }$ (numerical answers).

28. If $m\angle 1 = 42$, $m\angle 2 = x^2$, and $m\angle CED = 13x$, then $m\angle 2 = \underline{\ ?\ }$ or $m\angle 2 = \underline{\ ?\ }$ (numerical answers).



29. Given: $\square PQRS$; $\overline{PJ} \cong \overline{RK}$
Prove: $\overline{SJ} \cong \overline{QK}$

30. Given: $\square JQKS$; $\overline{PJ} \cong \overline{RK}$
Prove: $\angle P \cong \angle R$

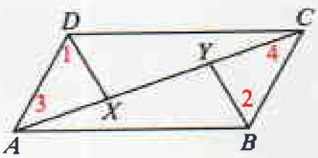


31. Given: $ABCD$ is a $\square$; $\overline{CD} \cong \overline{CE}$
Prove: $\angle A \cong \angle E$

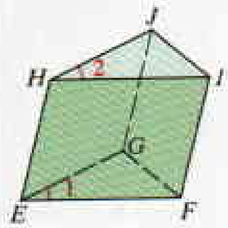
32. Given: $ABCD$ is a $\square$; $\angle A \cong \angle E$
Prove: $\overline{AB} \cong \overline{CE}$

Find something interesting to prove. Then prove it. Answers may vary.

33. Given: ▱$ABCD$; $\angle 1 \cong \angle 2$



34. Given: ▱$EFIH$; ▱$EGJH$; $\angle 1 \cong \angle 2$



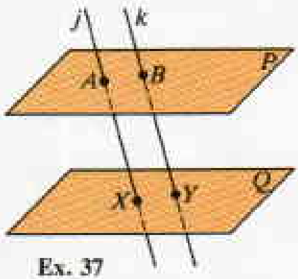
The coordinates of three vertices of a parallelogram are given. Find all the possibilities you can for the coordinates of the fourth vertex.

C **35.** (3, 4), (9, 4), (6, 8)

36. (−1, 0), (2, −2), (2, 2)

37. a. Given: Plane $P \parallel$ plane Q; $j \parallel k$
Prove: $AX = BY$

b. State a theorem about parallel planes and lines that you proved in part (a).

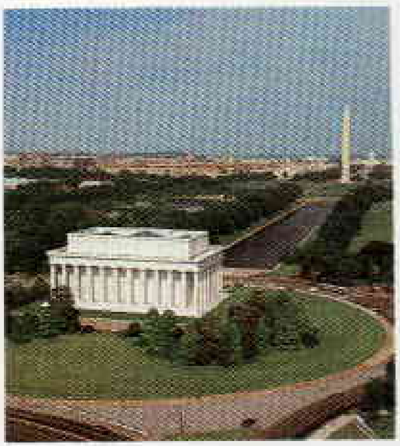


Ex. 37

38. Prove: If a segment whose endpoints lie on opposite sides of a parallelogram passes through the midpoint of a diagonal, that segment is bisected by the diagonal.

★ **39.** Write a paragraph proof: The sum of the lengths of the segments drawn from any point in the base of an isosceles triangle perpendicular to the legs is equal to the length of the altitude drawn to one leg.

Biographical Note — *Benjamin Banneker*



Benjamin Banneker (1731–1806) was a noted American scholar, largely self-taught, who became both a surveyor and an astronomer. As a surveyor, Banneker was a member of the commission that defined the boundary line and laid out the streets of the District of Columbia. As an astronomer, he accurately predicted a solar eclipse in 1789. From 1791 until his death he published almanacs containing information on astronomy, tide tables, and also such diverse subjects as insect life and medicinal products. Banneker's almanacs included ideas that were far ahead of their time, for example, the formation of a Department of the Interior and an organization like the United Nations.

5-2 *Ways to Prove that Quadrilaterals Are Parallelograms*

If both pairs of opposite sides of a quadrilateral are parallel, then by definition the quadrilateral is a parallelogram. The following theorems will give you additional ways to prove that a quadrilateral is a parallelogram.

Theorem 5-4

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: $\overline{TS} \cong \overline{QR}$; $\overline{TQ} \cong \overline{SR}$

Prove: Quad. $QRST$ is a ▱.

Plan for Proof: Draw $\overline{QS}$ and prove that $\triangle TSQ \cong \triangle RQS$. Then $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, and opposite sides are parallel.

Theorem 5-5

If one pair of opposite sides of a quadrilateral are both congruent and parallel, then the quadrilateral is a parallelogram.

Theorem 5-6

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Theorem 5-7

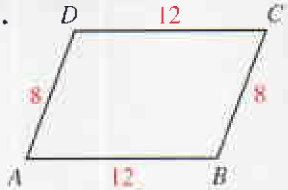
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

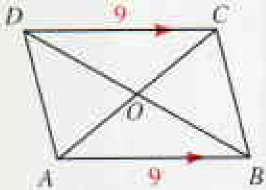
Five Ways to Prove that a Quadrilateral Is a Parallelogram

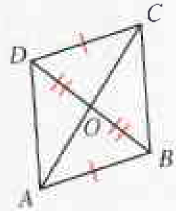
1. Show that *both* pairs of opposite sides are parallel.
2. Show that *both* pairs of opposite sides are congruent.
3. Show that *one* pair of opposite sides are both congruent and parallel.
4. Show that both pairs of opposite angles are congruent.
5. Show that the diagonals bisect each other.

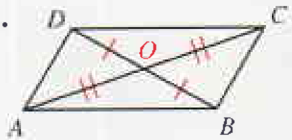
Classroom Exercises

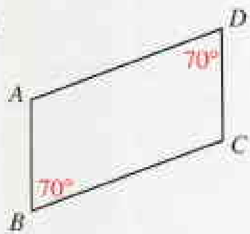
Study the markings on each figure and decide whether *ABCD must* be a parallelogram. If the answer is *yes*, state the definition or theorem that applies.

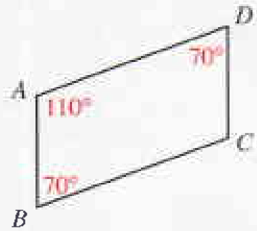
1. 

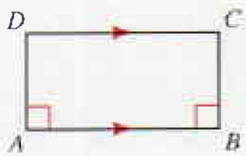
2. 

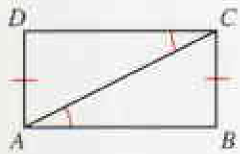
3. 

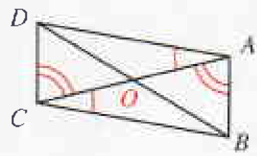
4. 

5. 

6. 

7. 

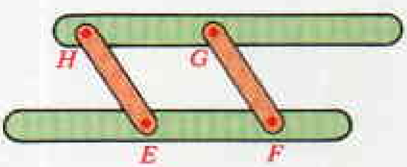
8. 

9. 

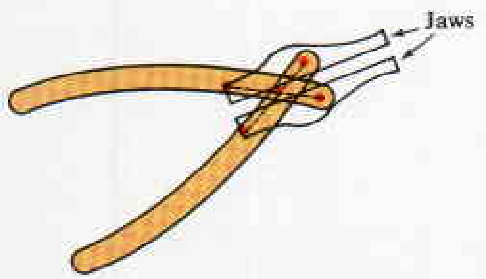
10. Draw a quadrilateral that has two pairs of congruent sides but that is *not* a parallelogram.

11. Draw a quadrilateral that is *not* a parallelogram but that has one pair of congruent sides and one pair of parallel sides.

12. *Parallel rulers*, used to draw parallel lines, are constructed so that $EF = HG$ and $HE = GF$. Since there are hinges at points E, F, G, and H, you can vary the distance between $\overleftrightarrow{HG}$ and $\overleftrightarrow{EF}$. Explain why $\overleftrightarrow{HG}$ and $\overleftrightarrow{EF}$ are always parallel.



13. The pliers shown are made in such a way that the jaws are always parallel. Explain.

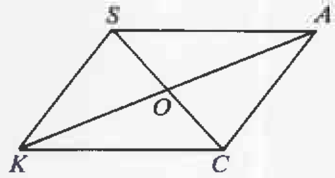


Written Exercises

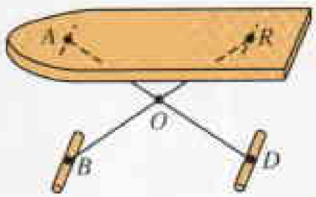
State the principal definition or theorem that enables you to deduce, from the information given, that quad. *SACK* is a parallelogram.

A

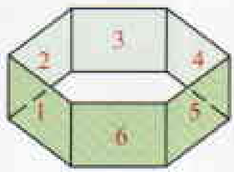
1. $\overline{SA} \parallel \overline{KC}$; $\overline{SK} \parallel \overline{AC}$
2. $\overline{SA} \cong \overline{KC}$; $\overline{SK} \cong \overline{AC}$
3. $\overline{SA} \cong \overline{KC}$; $\overline{SA} \parallel \overline{KC}$
4. $SO = \frac{1}{2}SC$; $KO = \frac{1}{2}KA$
5. $\angle SKC \cong \angle CAS$; $\angle KCA \cong \angle ASK$



6. Suppose you know that $\triangle SOK \cong \triangle COA$. Explain how you could prove that quad. *SACK* is a parallelogram.
7. The legs of this ironing board are built so that $BO = AO = RO = DO$. What theorem guarantees that the board is parallel to the floor ($\overline{AR} \parallel \overline{BD}$)?



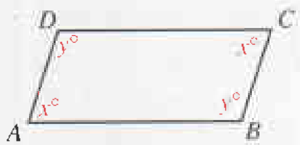
8. The quadrilaterals numbered 1, 2, 3, 4, and 5 are parallelograms. If you wanted to show that quadrilateral 6 is also a parallelogram, which of the five methods listed on page 172 would be easiest to use?



9. What theorem in this section is the converse of each theorem?
   a. Theorem 5-1 **b.** Theorem 5-2 **c.** Theorem 5-3
10. Give the reasons for each step in the following proof of Theorem 5-6.

Given: $m\angle A = m\angle C = x$;
$m\angle B = m\angle D = y$

Prove: *ABCD* is a ▱.



Proof:

Statements	Reasons
1. $m\angle A = m\angle C = x$; $m\angle B = m\angle D = y$	1. ?
2. $2x + 2y = 360$	2. ?
3. $x + y = 180$	3. ?
4. $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$	4. ?
5. *ABCD* is a ▱.	5. ?

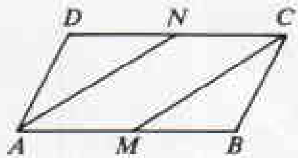
Draw and label a diagram. List what is given and what is to be proved. Then write a two-column proof of the theorem.

B **11.** Theorem 5-4 **12.** Theorem 5-5 **13.** Theorem 5-7

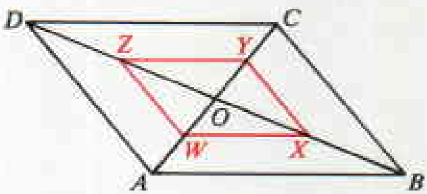
For Exercises 14–18 write paragraph proofs.

14. Given: ▱*ABCD*; *M* and *N* are the midpoints of $\overline{AB}$ and $\overline{DC}$.
Prove: *AMCN* is a ▱.

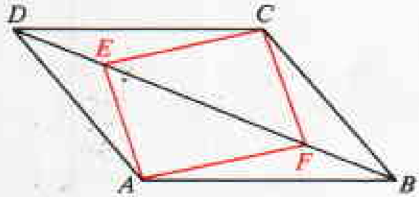
15. Given: ▱*ABCD*; $\overline{AN}$ bisects ∠*DAB*; $\overline{CM}$ bisects ∠*BCD*.
Prove: *AMCN* is a ▱.



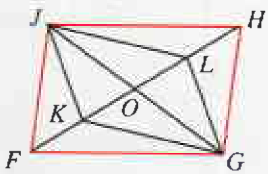
16. Given: ▱*ABCD*; *W*, *X*, *Y*, *Z* are midpoints of $\overline{AO}$, $\overline{BO}$, $\overline{CO}$, and $\overline{DO}$.
Prove: *WXYZ* is a ▱.



17. Given: ▱*ABCD*; *DE* = *BF*
Prove: *AFCE* is a ▱.



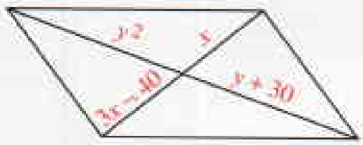
18. Given: ▱*KGLJ*; *FK* = *HL*
Prove: *FGHJ* is a ▱.



What values must *x* and *y* have to make the quadrilateral a parallelogram?

19.

20.



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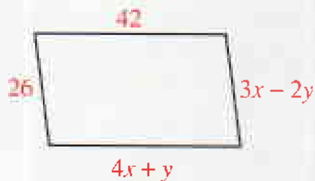
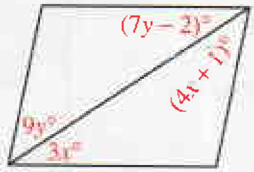
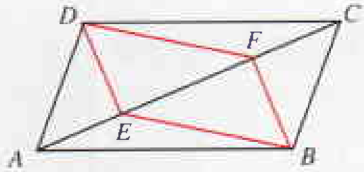


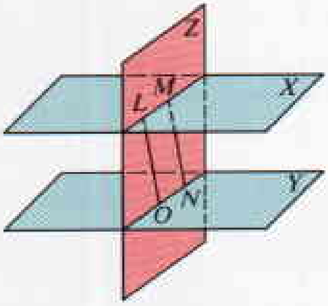
22.



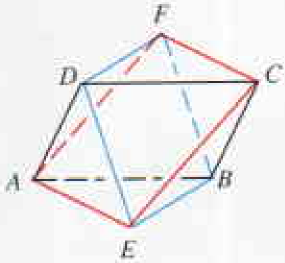
23. Given: ▱$ABCD$; $\overline{DE} \perp \overline{AC}$; $\overline{BF} \perp \overline{AC}$
Prove: $DEBF$ is a ▱.



24. Given: Plane $X \parallel$ plane Y; $\overline{LM} \cong \overline{ON}$
Prove: $LMNO$ is a ▱.



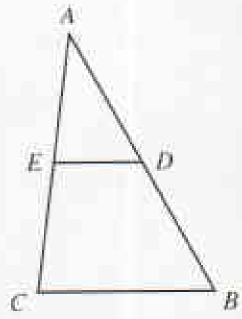
C 25. Write a paragraph proof.
Given: ▱$ABCD$; ▱$BEDF$
Prove: $AECF$ is a ▱.
(*Hint:* A short proof is possible if certain auxiliary segments are drawn.)



Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

Draw any $\triangle ABC$. Label the midpoint of $\overline{AB}$ as D. Draw a segment through D parallel to $\overline{BC}$ that intersects $\overline{AC}$ at E. Measure AE and EC. What do you notice?



Draw any $\triangle ABC$. Label the midpoints of $\overline{AB}$ and $\overline{AC}$ as D and E, respectively. Draw $\overline{DE}$. Measure $\angle AED$ and $\angle ACB$. What do you notice? What is true of $\overline{DE}$ and $\overline{BC}$? Measure DE and BC. What do you notice?

Write an equation that relates DE and BC. Repeat the drawing and measurements until you are sure of your equation.

5-3 *Theorems Involving Parallel Lines*

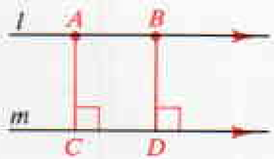
In this section we will prove four useful theorems about parallel lines. The first theorem uses the definition of the distance from a point to a line. (See page 154.)

Theorem 5-8

If two lines are parallel, then all points on one line are equidistant from the other line.

Given: $l \parallel m$; A and B are any points on l;
$\overline{AC} \perp m$; $\overline{BD} \perp m$

Prove: $AC = BD$



Proof:

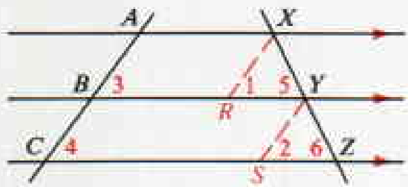
Since $\overline{AB}$ and $\overline{CD}$ are contained in parallel lines, $\overline{AB} \parallel \overline{CD}$. Since $\overline{AC}$ and $\overline{BD}$ are coplanar and are both perpendicular to m, they are parallel. Thus $ABDC$ is a parallelogram, by the definition of a parallelogram. Since opposite sides $\overline{AC}$ and $\overline{BD}$ are congruent, $AC = BD$.

Theorem 5-9

If three parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.

Given: $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ}$;
$\overline{AB} \cong \overline{BC}$

Prove: $\overline{XY} \cong \overline{YZ}$



Proof:

Through X and Y draw lines parallel to $\overleftrightarrow{AC}$, intersecting $\overleftrightarrow{BY}$ at R and $\overleftrightarrow{CZ}$ at S, as shown. Then $AXRB$ and $BYSC$ are parallelograms, by the definition of a parallelogram. Since the opposite sides of a parallelogram are congruent, $\overline{XR} \cong \overline{AB}$ and $\overline{BC} \cong \overline{YS}$. It is given that $\overline{AB} \cong \overline{BC}$, so using the Transitive Property twice gives $\overline{XR} \cong \overline{YS}$. Parallel lines are cut by transversals to form the following pairs of congruent corresponding angles:

$$\angle 1 \cong \angle 3 \qquad \angle 3 \cong \angle 4 \qquad \angle 4 \cong \angle 2 \qquad \angle 5 \cong \angle 6$$

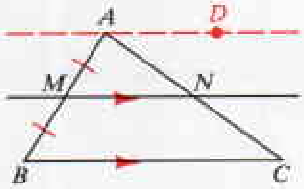
Then $\angle 1 \cong \angle 2$ (Transitive Property), and $\triangle XYR \cong \triangle YZS$ by AAS. Since $\overline{XY}$ and $\overline{YZ}$ are corresponding parts of these triangles, $\overline{XY} \cong \overline{YZ}$.

Theorem 5-10

A line that contains the midpoint of one side of a triangle and is parallel to another side passes through the midpoint of the third side.

Given: M is the midpoint of $\overline{AB}$;
$\overleftrightarrow{MN} \parallel \overline{BC}$

Prove: N is the midpoint of $\overline{AC}$.



Proof:

Let $\overleftrightarrow{AD}$ be the line through A parallel to $\overleftrightarrow{MN}$. Then $\overleftrightarrow{AD}$, $\overleftrightarrow{MN}$, and $\overleftrightarrow{BC}$ are three parallel lines that cut off congruent segments on transversal $\overleftrightarrow{AB}$. By Theorem 5-9 they also cut off congruent segments on $\overleftrightarrow{AC}$. Thus $\overline{AN} \cong \overline{NC}$ and N is the midpoint of $\overline{AC}$.

The next theorem has two parts, the first of which is closely related to Theorem 5-10.

Theorem 5-11

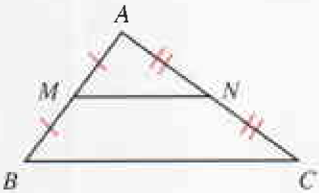
The segment that joins the midpoints of two sides of a triangle

(1) is parallel to the third side;

(2) is half as long as the third side.

Given: M is the midpoint of $\overline{AB}$;
N is the midpoint of $\overline{AC}$.

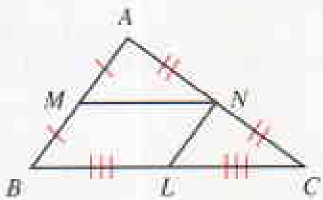
Prove: (1) $\overline{MN} \parallel \overline{BC}$
(2) $MN = \frac{1}{2}BC$



Proof of (1):

There is exactly one line through M parallel to $\overline{BC}$. By Theorem 5-10 that line passes through N, the midpoint of $\overline{AC}$. Thus $\overline{MN} \parallel \overline{BC}$.

Proof of (2):

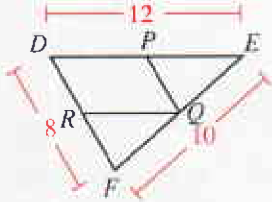


Let L be the midpoint of $\overline{BC}$, and draw $\overline{NL}$. By part (1), $\overline{MN} \parallel \overline{BC}$ and also $\overline{NL} \parallel \overline{AB}$. Thus quad. $MNLB$ is a parallelogram. Since its opposite sides are congruent, $MN = BL$. Since L is the midpoint of $\overline{BC}$, $BL = \frac{1}{2}BC$. Therefore $MN = \frac{1}{2}BC$.

Example P, Q, and R are midpoints of the sides of $\triangle DEF$.

a. What kind of figure is $DPQR$?

b. What is the perimeter of $DPQR$?

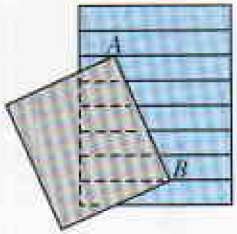


Solution **a.** Since $\overline{RQ} \parallel \overline{DE}$ and $\overline{PQ} \parallel \overline{DF}$, quad. $DPQR$ is a parallelogram.

b. $RQ = \frac{1}{2}DE = DP = 6$ and $PQ = \frac{1}{2}DF = DR = 4$. Thus the perimeter of $DPQR$ is $6 + 4 + 6 + 4$, or 20.

Classroom Exercises

1. You can use a sheet of lined notebook paper to divide a segment into a number of congruent parts. Here a piece of cardboard with edge $\overline{AB}$ is placed so that $\overline{AB}$ is separated into five congruent parts. Explain why this works.



***M*, *N*, and *T* are the midpoints of the sides of $\triangle XYZ$.**

2. If $XZ = 10$, then $MN =$ _?_.

3. If $TN = 7$, then $XY =$ _?_.

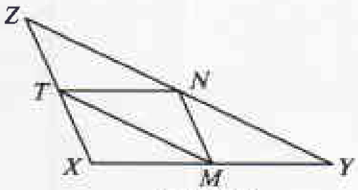
4. If $ZN = 8$, then $TM =$ _?_.

5. If $XY = k$, then $TN =$ _?_.

6. Suppose $XY = 10$, $YZ = 14$, and $XZ = 8$. What are the lengths of the three sides of

a. $\triangle TNZ$? **b.** $\triangle MYN$?

c. $\triangle XMT$? **d.** $\triangle NTM$?



Exs. 2–8

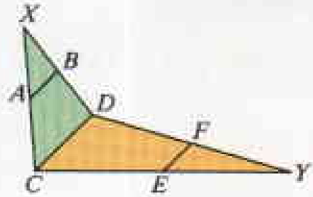
7. State a theorem suggested by Exercise 6.

8. How many parallelograms are in the diagram?

9. What result of this section do the railings suggest?

Written Exercises

Points A, B, E, and F are the midpoints of $\overline{XC}$, $\overline{XD}$, $\overline{YC}$, and $\overline{YD}$. Complete.



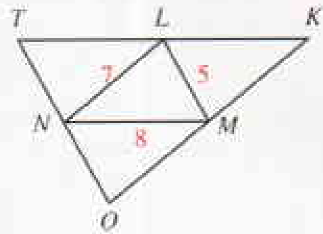
A 1. If $CD = 24$, then $AB =$ _?_ and $EF =$ _?_.

2. If $AB = k$, then $CD =$ _?_ and $EF =$ _?_.

3. If $AB = 5x - 8$ and $EF = 3x$, then $x =$ _?_.

4. If $CD = 8x$ and $AB = 3x + 2$, then $x =$ _?_.

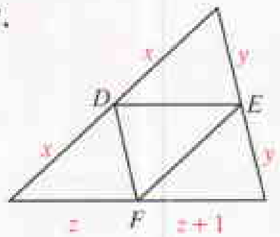
5. Given: L, M, and N are midpoints of the sides of $\triangle TKO$. Find the perimeter of each figure.
   a. $\triangle TKO$ b. $\triangle LMK$
   c. $\square TNML$ d. quad. $LNOK$



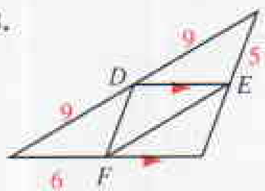
6. a. Name all triangles congruent to $\triangle TNL$.
   b. Suppose you are told that the area of $\triangle NLM$ is 17.32 cm^2. What is the area of $\triangle TKO$?

Name all the points shown that *must* be midpoints of the sides of the large triangle.

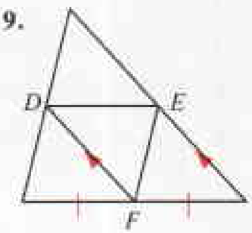
7.



8.



9.



$\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CG}$, and $\overleftrightarrow{DH}$ are parallel, with $EF = FG = GH$. Complete.

10. If $AB = 5$, then $AD =$ _?_.

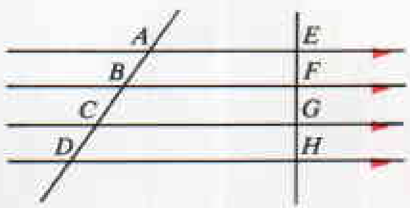
11. If $AC = 12$, then $CD =$ _?_.

12. If $AB = 5x$ and $BC = 2x + 12$, then $x =$ _?_.

13. If $AC = 22 - x$ and $BD = 3x - 22$, then $x =$ _?_.

B 14. If $AB = 15$, $BC = 2x - y$, and $CD = x + y$, then $x =$ _?_ and $y =$ _?_.

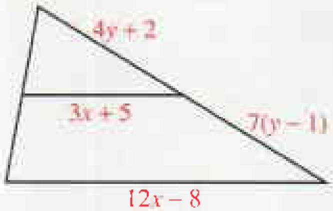
15. If $AB = 12$, $BC = 2x + 3y$, and $BD = 8x$, then $x =$ _?_ and $y =$ _?_.



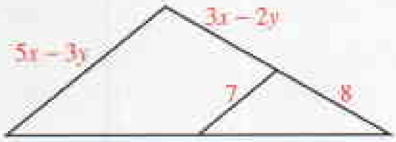
Exs. 10–15

In Exercises 16–17 a segment joins the midpoints of two sides of a triangle. Find the values of x and y.

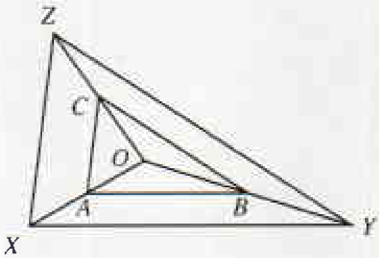
16.



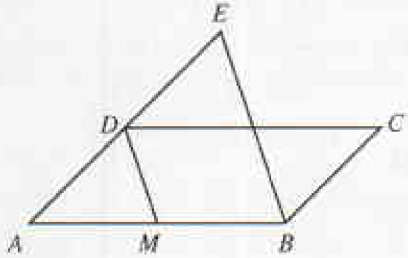
17.



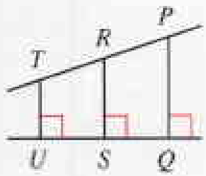
18. Given: A is the midpoint of $\overline{OX}$; $\overline{AB} \parallel \overline{XY}$; $\overline{BC} \parallel \overline{YZ}$
Prove: $\overline{AC} \parallel \overline{XZ}$



19. Given: $\square ABCD$; $\overline{BE} \parallel \overline{MD}$; M is the midpoint of $\overline{AB}$.
Prove: $DE = BC$

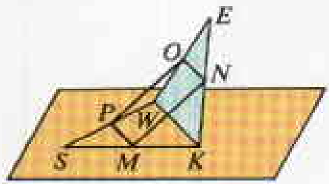


20. Given: $\overline{PQ}$, $\overline{RS}$, and $\overline{TU}$ are each perpendicular to $\overleftrightarrow{UQ}$; R is the midpoint of $\overline{PT}$.
Prove: R is equidistant from U and Q.



21. $EFGH$ is a parallelogram whose diagonals intersect at P. M is the midpoint of $\overline{FG}$. Prove that $MP = \frac{1}{2}EF$.

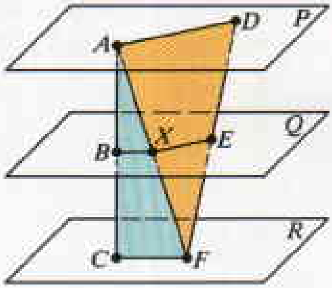
22. A *skew quadrilateral SKEW* is shown. M, N, O, and P are the midpoints of $\overline{SK}$, $\overline{KE}$, $\overline{WE}$, and $\overline{SW}$. Explain why $PMNO$ is a parallelogram.



23. Draw $\triangle ABC$ and label the midpoints of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ as X, Y, and Z, respectively. Let P be the midpoint of $\overline{BZ}$ and Q be the midpoint of $\overline{CZ}$. Prove that $PX = QY$.

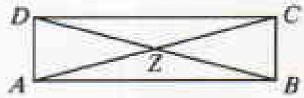
24. Draw $\triangle ABC$ and let D be the midpoint of $\overline{AB}$. Let E be the midpoint of $\overline{CD}$. Let F be the intersection of $\overrightarrow{AE}$ and $\overline{BC}$. Draw $\overline{DG}$ parallel to $\overline{EF}$ meeting $\overline{BC}$ at G. Prove that $BG = GF = FC$.

C **25.** Given: Parallel planes P, Q, and R cutting transversals $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$; $AB = BC$
Prove: $DE = EF$
(*Hint:* You can't assume that $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ are coplanar. Draw $\overline{AF}$, cutting plane Q at X. Using the plane of $\overline{AC}$ and $\overline{AF}$, apply Theorems 3-1 and 5-10. Then use the plane of $\overline{AF}$ and $\overline{FD}$.)



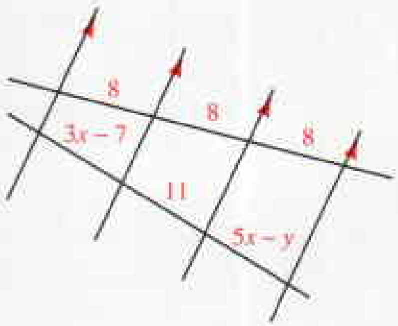
Self-Test 1

The diagonals of ▱*ABCD* intersect at *Z*. Tell whether each statement *must be*, *may be*, or *cannot be* true.

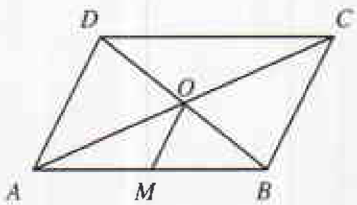


1. $\overline{AC} \cong \overline{BD}$ **2.** $\overline{DZ} \cong \overline{BZ}$
3. $\overline{AD} \parallel \overline{BC}$ **4.** $m\angle DAB = 85$ and $m\angle BCD = 95$
5. List five ways to prove that quad. *ABCD* is a parallelogram.

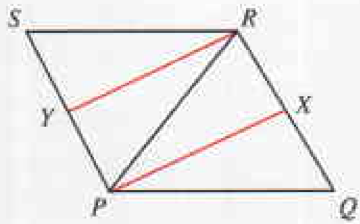
6. a. State a theorem that allows you to conclude that $3x - 7 = 11$.
b. Find the values of x and y.



7. Given: ▱*ABCD*;
M is the midpoint of $\overline{AB}$.
Prove: $MO = \frac{1}{2}AD$



8. Given: ▱*PQRS*;
$\overline{PX}$ bisects $\angle QPR$;
$\overline{RY}$ bisects $\angle SRP$.
Prove: *RYPX* is a ▱.



♦ Computer Key-In

The BASIC computer program below will calculate the lengths of sides and diagonals of a quadrilateral. (Line 200 uses the distance formula, which you will study in Chapter 13.) Do the exercises to see what you can discover before studying special quadrilaterals in the next two sections.

```
10  DIM X(4), Y(4), A$(4)
15  A$(1) = "A":A$(2) = "B":A$(3) = "C":A$(4) = "D"
20  FOR I = 1 TO 4
30  PRINT "VERTEX ";A$(I);" ";
40  INPUT X(I), Y(I)
50  NEXT I
60  PRINT "SIDES"
70  FOR I = 1 TO 4
80  LET J = I + 1 - 4 * INT(I/4)
90  GOSUB 200
95  NEXT I
100  PRINT "DIAGONALS"
110  LET I = 1:LET J = 3
120  GOSUB 200
130  LET I = 2:LET J = 4
140  GOSUB 200
150  END
200  LET D = SQR((X(I)-X(J))↑2 + (Y(I)-Y(J))↑2)
210  LET D = INT(100 * D + .5)/100
220  PRINT A$(I);A$(J);" = ";D
230  RETURN
```

Exercises

Plot the given points and draw quad. *ABCD* and its diagonals. Also RUN the program above, inputting the given coordinates for the vertices. Then tell which of the following statements are true for quad. *ABCD*.

I. Quad. *ABCD* is a parallelogram.
II. Both pairs of opposite sides are congruent.
III. All sides are congruent.
IV. The diagonals are congruent.
V. The diagonals are perpendicular.
VI. The sides form four right angles.

1. a. $A(1, 1)$, $B(3, 4)$, $C(10, 4)$, $D(8, 1)$
b. $A(0, -3)$, $B(-4, -1)$, $C(-2, 1)$, $D(2, -1)$
2. a. $A(1, 2)$, $B(-3, -1)$, $C(-7, 2)$, $D(-3, 5)$
b. $A(1, 1)$, $B(3, -3)$, $C(-1, -1)$, $D(-3, 3)$
3. a. $A(4, -1)$, $B(-2, -1)$, $C(-2, 2)$, $D(4, 2)$
b. $A(-5, 0)$, $B(-3, 4)$, $C(5, 0)$, $D(3, -4)$
4. a. $A(2, 2)$, $B(2, -2)$, $C(-2, -2)$, $D(-2, 2)$
b. $A(0, 3)$, $B(3, 0)$, $C(0, -3)$, $D(-3, 0)$
5. a. $A(-7, 0)$, $B(-4, 4)$, $C(-1, 4)$, $D(-1, 0)$
b. $A(4, 6)$, $B(10, 3)$, $C(4, -3)$, $D(1, 3)$
6. a. $A(0, 0)$, $B(2, 4)$, $C(4, 0)$, $D(2, -2)$
b. $A(6, 0)$, $B(3, -5)$, $C(-4, 0)$, $D(3, 5)$

Special Quadrilaterals

Objectives

1. Apply the definitions and identify the special properties of a rectangle, a rhombus, and a square.
2. Determine when a parallelogram is a rectangle, rhombus, or square.
3. Apply the definitions and identify the properties of a trapezoid and an isosceles trapezoid.

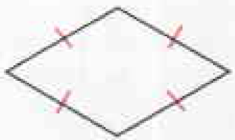
5-4 *Special Parallelograms*

In this section you will study the properties of special parallelograms: *rectangles*, *rhombuses*, and *squares*.

A **rectangle** is a quadrilateral with four right angles. Therefore, every rectangle is a parallelogram. (Why?)

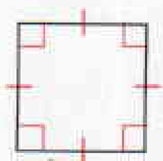
Rectangle

A **rhombus** is a quadrilateral with four congruent sides. Therefore, every rhombus is a parallelogram. (Why?)



Rhombus

A **square** is a quadrilateral with four right angles and four congruent sides. Therefore, every square is a rectangle, a rhombus, and a parallelogram. (Why?)



Square

Since rectangles, rhombuses, and squares are parallelograms, they have all the properties of parallelograms. They also have the special properties given in the theorems on the next page. Proofs of these theorems are left as exercises.

Theorem 5-12

The diagonals of a rectangle are congruent.

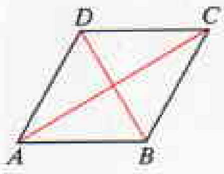
Theorem 5-13

The diagonals of a rhombus are perpendicular.

Theorem 5-14

Each diagonal of a rhombus bisects two angles of the rhombus.

Example Given: $ABCD$ is a rhombus.
What can you conclude?

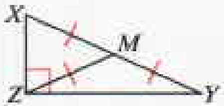


Solution $ABCD$ is a parallelogram, with all the properties of a parallelogram. Also:
By Theorem 5-13, $\overline{AC} \perp \overline{BD}$.
By Theorem 5-14, $\overline{AC}$ bisects $\angle DAB$ and $\angle BCD$;
$\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

The properties of rectangles lead to the following interesting conclusion about any right triangle.

Begin with rt. $\triangle XYZ$.

1. Draw lines to form rectangle $XZYK$. (How?)
2. Draw $\overline{ZK}$. $ZK = XY$ (Why?)
3. $\overline{ZK}$ and $\overline{XY}$ bisect each other. (Why?)
4. $MX = MY = MZ = MK$, by (2) and (3).



Since $MX = MY = MZ$, we have shown the following.

Theorem 5-15

The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Proofs of the next two theorems will be discussed as Classroom Exercises.

Theorem 5-16

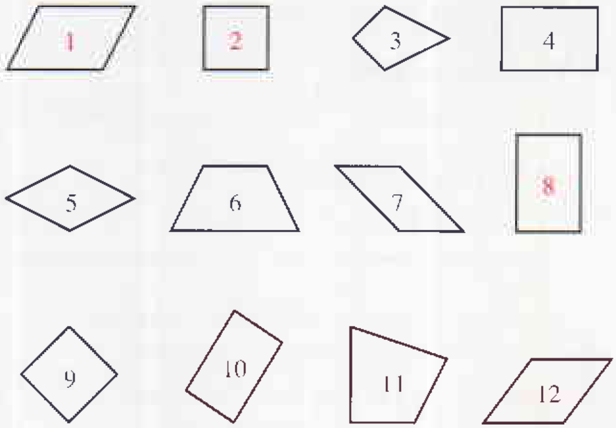
If an angle of a parallelogram is a right angle, then the parallelogram is a rectangle.

Theorem 5-17

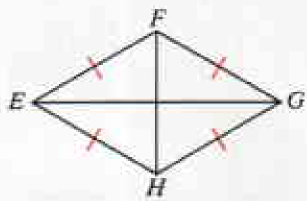
If two consecutive sides of a parallelogram are congruent, then the parallelogram is a rhombus.

Classroom Exercises

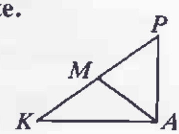
1. Name each figure shown that appears to be:
   a. a parallelogram
   b. a rectangle
   c. a rhombus
   d. a square
2. Name each figure that is *both* a rectangle *and* a rhombus.
3. Name each figure that is a rectangle but not a square.
4. Name each figure that is a rhombus but not a square.



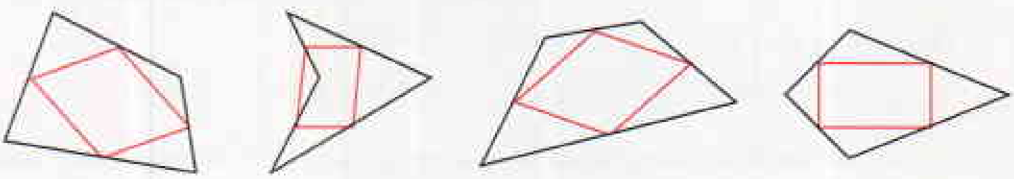
5. When you know that one angle of a parallelogram is a right angle, you can prove that the parallelogram is a rectangle. Draw a diagram and explain.
6. When you know that two consecutive sides of a parallelogram are congruent, you can prove that the parallelogram is a rhombus. Draw a diagram and explain.
7. Given: Rhombus *EFGH*
   a. *F*, being equidistant from *E* and *G*, must lie on the _?_ of $\overline{EG}$.
   b. *H*, being equidistant from *E* and *G*, must lie on the _?_ of $\overline{EG}$.
   c. From (a) and (b) you can deduce that $\overline{FH}$ is the _?_ of $\overline{EG}$.
   d. State the theorem of this section that you have just proved.



$\angle KAP$ is a right angle, and $\overline{AM}$ is a median. Complete.



8. If $MP = 6\frac{1}{2}$, then $MA =$ _?_.
9. If $MA = t$, then $KP =$ _?_.
10. If $m\angle K = 40$, then $m\angle KAM =$ _?_.
11. In the diagrams below, the red figures are formed by joining the midpoints of the sides of the quadrilaterals.
    a. What seems to be the common property of the red figures?
    b. Describe how you would prove your answer to part (a).

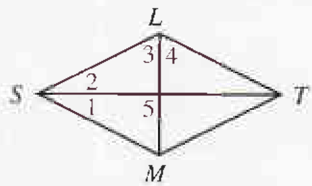


Written Exercises

Copy the chart. Then place check marks in the appropriate spaces.

	Property	Parallelogram	Rectangle	Rhombus	Square
A **1.**	Opp. sides are ∥.				
2.	Opp. sides are ≅.				
3.	Opp. ∠s are ≅.				
4.	A diag. forms two ≅ △s.				
5.	Diags. bisect each other.				
6.	Diags. are ≅.				
7.	Diags. are ⊥.				
8.	A diag. bisects two ∠s.				
9.	All ∠s are rt. ∠s.				
10.	All sides are ≅.				

Quad. *SLTM* is a rhombus.

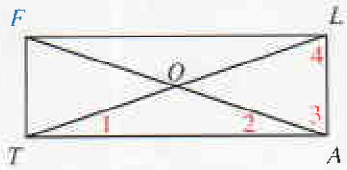


11. If $m\angle 1 = 25$, find the measures of $\angle 2$, $\angle 3$, $\angle 4$, and $\angle 5$.

12. If $m\angle 1 = 3x + 8$ and $m\angle 2 = 11x - 24$, find the value of x.

13. If $m\angle 1 = 3x + 1$ and $m\angle 3 = 7x - 11$, find the value of x.

Quad. *FLAT* is a rectangle.

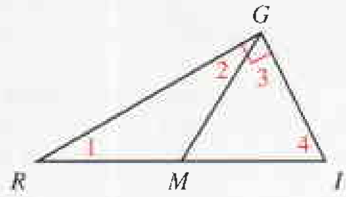


14. If $m\angle 1 = 18$, find the measures of $\angle 2$, $\angle 3$, and $\angle 4$.

15. If $FA = 27$, find LO.

16. If $TO = 4y + 7$ and $FA = 30$, find the value of y.

$\overline{GM}$ is a median of right $\triangle IRG$.



17. If $m\angle 1 = 32$, find the measures of $\angle 2$, $\angle 3$, and $\angle 4$.

18. If $m\angle 4 = 7x - 3$ and $m\angle 3 = 6(x + 1)$, find the value of x.

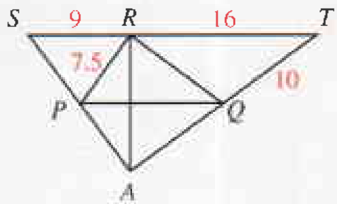
19. If $GM = 2y + 3$ and $RI = 12 - 8y$, find the value of y.

The coordinates of three vertices of a rectangle are given. Plot the points and find the coordinates of the fourth vertex. Is the rectangle a square?

20. $O(0, 0)$, $P(0, 5)$, $Q(\underline{\ ?\ }, \underline{\ ?\ })$, $R(2, 0)$ **21.** $A(2, 1)$, $B(4, 1)$, $C(4, 5)$, $D(\underline{\ ?\ }, \underline{\ ?\ })$

22. $O(0, 0)$, $E(4, 0)$, $F(4, 3)$, $G(\underline{\ ?\ }, \underline{\ ?\ })$ **23.** $H(1, 3)$, $I(4, 3)$, $J(\underline{\ ?\ }, \underline{\ ?\ })$, $K(1, 6)$

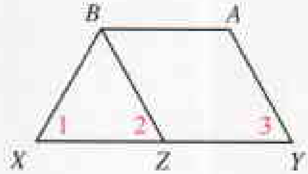
$\overline{RA}$ is an altitude of $\triangle SAT$. P and Q are midpoints of $\overline{SA}$ and $\overline{TA}$. $SR = 9$, $RT = 16$, $QT = 10$, and $PR = 7.5$.



B **24.** Find RQ. **25.** Find SA.

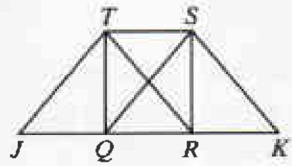
26. Find the perimeter of $\triangle PQR$.

27. Find the perimeter of $\triangle SAT$.



28. Given: ▱$ABZY$; $\overline{ZY} \cong \overline{BX}$; $\angle 1 \cong \angle 2$
Prove: $ABZY$ is a rhombus.

29. Given: ▱$ABZY$; $\overline{AY} \cong \overline{BX}$
Prove: $\angle 1 \cong \angle 2$ and $\angle 1 \cong \angle 3$



30. Given: Rectangle $QRST$; ▱$RKST$
Prove: $\triangle QSK$ is isosceles.

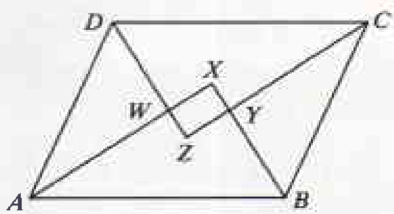
31. Given: Rectangle $QRST$; ▱$RKST$; ▱$JQST$
Prove: $\overline{JT} \cong \overline{KS}$

32. Prove Theorem 5-12.

33. Prove Theorem 5-14 for one diagonal of the rhombus. (Note that a proof for the other would be similar, step-by-step.)

34. Prove: If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

35. Prove: If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.



36. **a.** The bisectors of the angles of ▱$ABCD$ intersect to form quad. $WXYZ$. What special kind of quadrilateral is $WXYZ$?
b. Prove your answer to part (a).

37. Draw a rectangle and bisect its angles. The bisectors intersect to form what special kind of quadrilateral?

The coordinates of three vertices of a rhombus are given, not necessarily in order. Plot the points and find the coordinates of the fourth vertex. Measure the sides to check your answer.

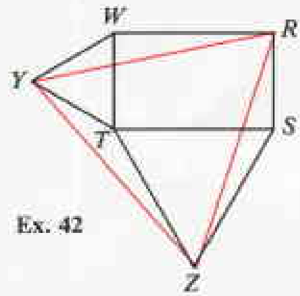
38. $O(0, 0)$, $L(5, 0)$, $D(4, 3)$, $V(\underline{\ ?\ }, \underline{\ ?\ })$ **39.** $O(0, 0)$, $S(0, 10)$, $E(6, 18)$, $W(\underline{\ ?\ }, \underline{\ ?\ })$

C **40. a.** Suppose that two sides of a quadrilateral are parallel and that one diagonal bisects an angle. Does that quadrilateral have to be special in other ways? If so, write a proof. If not, draw a convincing diagram.

b. Repeat part (a) with these conditions: Suppose that two sides are parallel and that one diagonal bisects two angles of the quadrilateral.

41. Draw a regular pentagon *ABCDE*. Let *X* be the intersection of $\overline{AC}$ and $\overline{BD}$. What special kind of quadrilateral is *AXDE*? Write a paragraph proof.

42. Given: Rectangle *RSTW*;
equilateral △ *YWT* and *STZ*
What is true of △*RYZ*?
Write a paragraph proof.



Ex. 42

Explorations

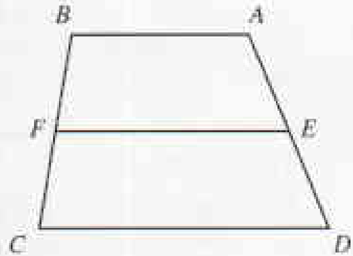
These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

As you will learn in the next section, a *trapezoid* is a quadrilateral with exactly one pair of parallel sides.

Draw trapezoid *ABCD* with $\overline{BA} \parallel \overline{CD}$. Label the midpoints of $\overline{AD}$ and $\overline{BC}$ as *E* and *F* respectively, and draw $\overline{FE}$.

Measure ∠*BFE* and ∠*BCD*. What is true of $\overline{CD}$ and $\overline{FE}$? What postulate or theorem tells you this?

What is true of $\overline{FE}$ and $\overline{BA}$? Why?



Measure the lengths of $\overline{BA}$, $\overline{CD}$, and $\overline{FE}$. What do you notice?

Write an equation that relates *BA*, *CD*, and *FE*. Repeat the drawing and measurements until you are sure of your equation.

Mixed Review Exercises

Find the average of the given numbers. (The *average* is the sum of the numbers divided by the number of numbers.)

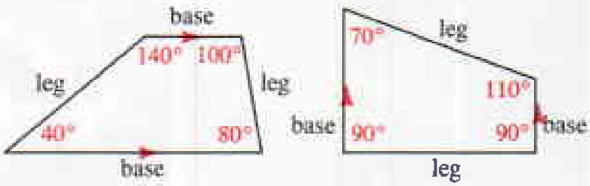
1. 17, 9 **2.** 15, 25 **3.** 18, 2, 13 **4.** 7, 8, 5, 15, 10

5. 7.9, 8.5 **6.** 4, −7 **7.** −3, 4, −7, 10 **8.** 1.7, 2.6, 9.1, 0.4

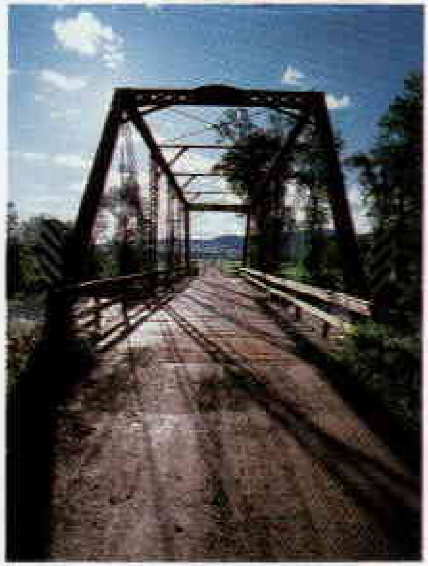
9. The numbers given are the coordinates of the endpoints of a segment on a number line. Find the coordinate of the midpoint by taking the average.
a. 12, 34 **b.** −3, 7 **c.** 17, −9 **d.** −5, −7

5-5 *Trapezoids*

A quadrilateral with exactly one pair of parallel sides is called a **trapezoid**. The parallel sides are called the **bases**. The other sides are **legs**.



Trapezoids



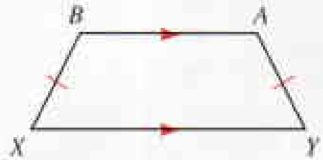
A trapezoid with congruent legs is called an **isosceles trapezoid**. If you fold any isosceles trapezoid so that the legs coincide, you will find that both pairs of *base angles* are congruent.

Isosceles trapezoid

A trapezoidal shape can be seen in the photograph. Is it isosceles?

Theorem 5-18

Base angles of an isosceles trapezoid are congruent.

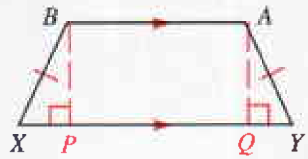


Given: Trapezoid $ABXY$ with $\overline{BX} \cong \overline{AY}$

Prove: $\angle X \cong \angle Y$; $\angle B \cong \angle A$

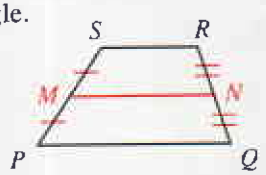
Plan for Proof: Note that the diagram does not contain any parallelograms or congruent triangles that might be used. To obtain such figures, you could draw auxiliary lines. You could use either diagram below to prove the theorem.

Draw $\overline{BZ} \parallel \overline{AY}$ so that $ABZY$ is a $\square$.

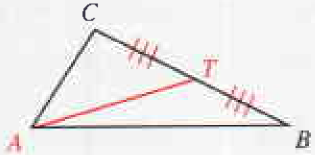


Draw $\overline{BP} \perp \overline{XY}$ and $\overline{AQ} \perp \overline{XY}$.
Since $\overline{BA} \parallel \overline{XY}$, $BP = AQ$ by Theorem 5-8.

The **median** of a trapezoid is the segment that joins the midpoints of the legs. Note the difference between the median of a trapezoid and a median of a triangle.



$\overline{MN}$ is *the* median of trapezoid $PQRS$.



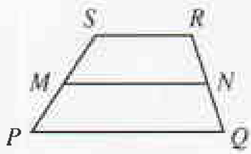
$\overline{AT}$ is *a* median of $\triangle ABC$.

Theorem 5-19

The median of a trapezoid

(1) is parallel to the bases;

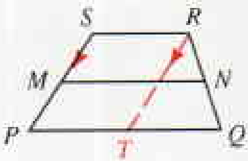
(2) has a length equal to the average of the base lengths.



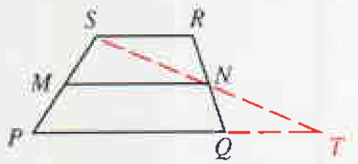
Given: Trapezoid $PQRS$ with median $\overline{MN}$

Prove: (1) $\overline{MN} \parallel \overline{PQ}$ and $\overline{MN} \parallel \overline{SR}$ (2) $MN = \frac{1}{2}(PQ + SR)$

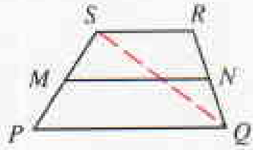
Plan for Proof: Again it is necessary to introduce auxiliary lines, and again there is more than one way to do this. Although any of the diagrams below could be used to prove the theorem, the proof below uses the second diagram.



Draw $\overline{RT} \parallel \overline{SP}$.



Draw $\overrightarrow{SN}$ intersecting $\overrightarrow{PQ}$ at T.

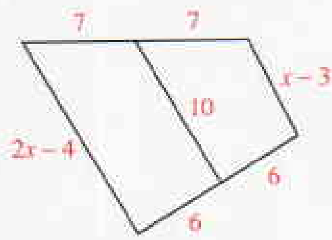


Draw $\overline{SQ}$.

Proof:

Extend $\overline{SN}$ to intersect $\overrightarrow{PQ}$ at T. $\triangle TNQ \cong \triangle SNR$ by ASA. Then $SN = NT$ and N is the midpoint of $\overline{ST}$. Using Theorem 5-11 and $\triangle PST$, (1) $\overline{MN} \parallel \overline{PQ}$ (and also $\overline{MN} \parallel \overline{SR}$), and (2) $MN = \frac{1}{2}PT = \frac{1}{2}(PQ + QT) = \frac{1}{2}(PQ + SR)$, since $\overline{QT} \cong \overline{SR}$.

Example A trapezoid and its median are shown. Find the value of x.

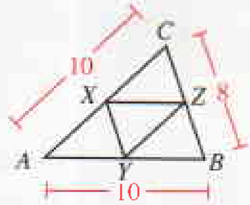


Solution

$$10 = \frac{1}{2}[(2x - 4) + (x - 3)]$$
$$20 = (2x - 4) + (x - 3)$$
$$20 = 3x - 7$$
$$27 = 3x$$
$$9 = x$$

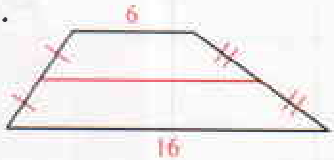
Classroom Exercises

X, Y, and Z are midpoints of the sides of isosceles $\triangle ABC$.



1. Explain why $XZBA$ is a trapezoid.
2. Name two trapezoids other than $XZBA$.
3. Name an isosceles trapezoid and find its perimeter.

Find the length of the median of each trapezoid.

4.


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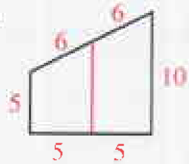

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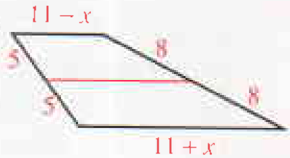


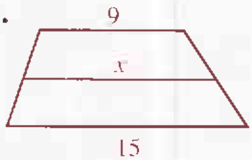
Draw the trapezoid described. If such a trapezoid cannot be drawn, explain why not.

7. with two right angles
8. with both bases shorter than the legs
9. with congruent bases
10. with three acute angles
11. Draw a quadrilateral such that exactly three sides are congruent and two pairs of angles are congruent.

Written Exercises

Each diagram shows a trapezoid and its median. Find the value of x.

A

1.


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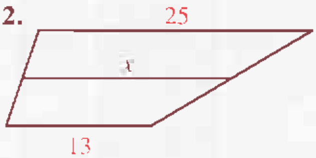

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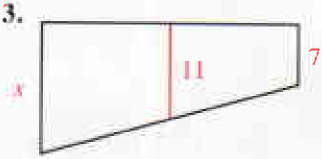

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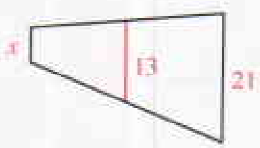

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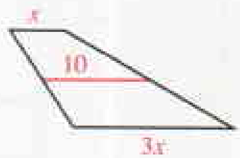

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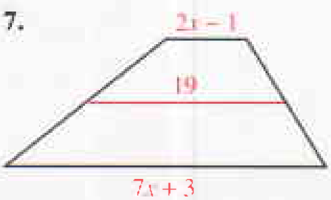
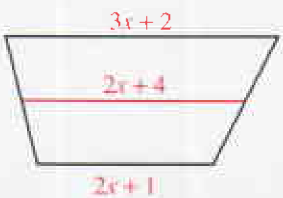


8.


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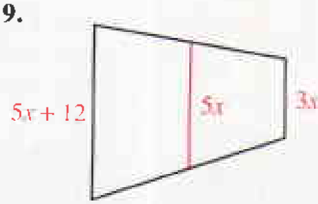
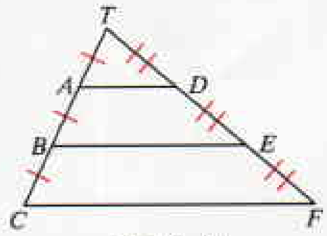


10. One angle of an isosceles trapezoid has measure 57. Find the measures of the other angles.

11. Two congruent angles of an isosceles trapezoid have measures $3x + 10$ and $5x - 10$. Find the value of x and then give the measures of all angles of the trapezoid.

In Exercises 12–20, $TA = AB = BC$ and $TD = DE = EF$.



Exs. 12–20

12. Write an equation that relates AD and BE. (*Hint:* Think of $\triangle TBE$.)

13. Write an equation that relates AD, BE, and CF. (*Hint:* Think of trapezoid $CFDA$.)

14. If $AD = 7$, then $BE =$ __?__ and $CF =$ __?__.

15. If $BE = 26$, then $AD =$ __?__ and $CF =$ __?__.

16. If $AD = x$ and $BE = x + 6$, then $x =$ __?__ and $CF =$ __?__ (numerical answers).

B

17. If $AD = x + 3$, $BE = x + y$, and $CF = 36$, then $x =$ __?__ and $y =$ __?__.

18. If $AD = x + y$, $BE = 20$, and $CF = 4x - y$, then $CF =$ __?__ (numerical answer).

19. Tony makes up a problem for the figure, setting $AD = 5$ and $CF = 17$. Katie says, ''You can't do that.'' Explain.

20. Mike makes up a problem for the figure, setting $AD = 2x + 1$, $BE = 4x + 2$, and $CF = 6x + 3$ and asking for the value of x. Katie says, ''Anybody can do that problem.'' Explain.

Draw a quadrilateral of the type named. Join, in order, the midpoints of the sides. What special kind of quadrilateral do you appear to get?

21. rhombus

22. rectangle

23. isosceles trapezoid

24. non-isosceles trapezoid

25. quadrilateral with no congruent sides

26. Carefully draw an isosceles trapezoid and measure its diagonals. What do you discover? Write a proof of your discovery.

27. Prove Theorem 5-18.

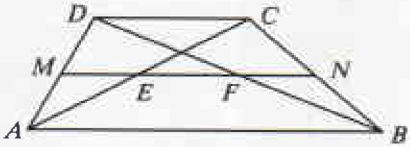
A *kite* is a quadrilateral that has two pairs of congruent sides, but opposite sides are not congruent.

28. Draw a convex kite. Discover, state, and prove whatever you can about the diagonals and angles of a kite.

29. a. Draw a convex kite. Join, in order, the midpoints of the sides. What special kind of quadrilateral do you appear to get?

b. Repeat part (a), but draw a nonconvex kite.

$ABCD$ is a trapezoid with median $\overline{MN}$.

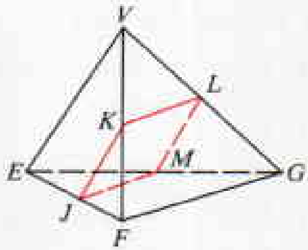


30. If $DC = 6$ and $AB = 16$, find ME, FN, and EF.

31. Prove that $EF = \frac{1}{2}(AB - DC)$.

32. If $DC = 3x$, $AB = 2x^2$, and $EF = 7$, find the value of x.

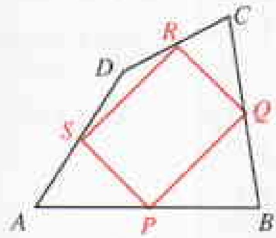
C **33.** $\overline{VE}$ and $\overline{FG}$ are congruent. J, K, L, and M are the midpoints of $\overline{EF}$, $\overline{VF}$, $\overline{VG}$, and $\overline{EG}$. What name best describes $JKLM$? Explain.



34. When the midpoints of the sides of quad. $ABCD$ are joined, rectangle $PQRS$ is formed.

a. Draw other quadrilaterals $ABCD$ with this property.

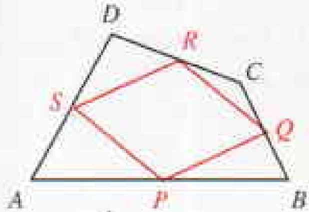
b. What must be true of quad. $ABCD$ if $PQRS$ is to be a rectangle?



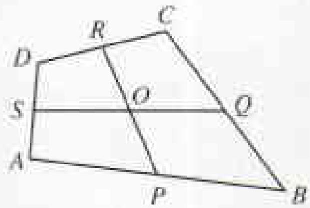
35. When the midpoints of the quad. $ABCD$ are joined, rhombus $PQRS$ is formed.

a. Draw other quadrilaterals $ABCD$ with this property.

b. What must be true of quad. $ABCD$ if $PQRS$ is to be a rhombus?

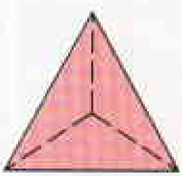


36. P, Q, R, and S are the midpoints of the sides of quad. $ABCD$. In this diagram $\overline{PR}$ and $\overline{SQ}$ have the same midpoint, point O. If you think this will be the case for *any* quad. $ABCD$, prove it. If not, tell what other information you need to know about quad. $ABCD$ before you can conclude that $\overline{PR}$ and $\overline{SQ}$ have the same midpoint.



Challenge

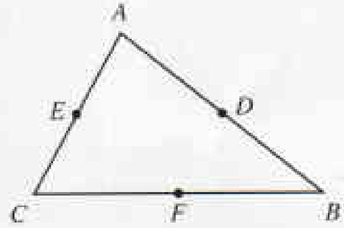
The three-dimensional figure shown has six congruent edges. Draw four such figures. On your diagrams show how a plane can intersect the figure to form **(a)** a triangle with three congruent sides, **(b)** a triangle with sides not all congruent, **(c)** a rectangle, and **(d)** an isosceles trapezoid.



Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

Draw any $\triangle ABC$. Label the midpoint of $\overline{AB}$ as D, of $\overline{AC}$ as E, and of $\overline{BC}$ as F.



Form a quadrilateral ($ABFE$, $BCED$, or $CADF$) by using two midpoints and two vertices.

What kind of quadrilateral is each of $ABFE$, $BCED$, and $CADF$? How do you know?

Form a quadrilateral ($ADFE$, $BFED$, or $CEDF$) by using three midpoints and a vertex.

What kind of quadrilateral is each of $ADFE$, $BFED$, and $CEDF$? How do you know?

Self-Test 2

Quad. *WXYZ* must be a special figure to meet the conditions stated. Write the best name for that special quadrilateral.

1. $\overline{WX} \cong \overline{YZ}$ and $\overline{WX} \parallel \overline{YZ}$

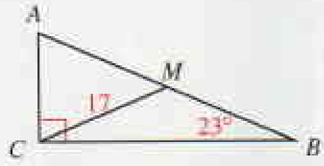
2. $\overline{WX} \parallel \overline{YZ}$ and $\overline{WX} \not\cong \overline{YZ}$

3. $\overline{WX} \cong \overline{YZ}$, $\overline{XY} \cong \overline{ZW}$, and diag. $\overline{WY} \cong$ diag. $\overline{XZ}$

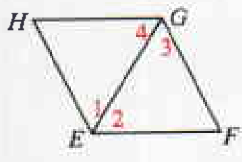
4. Diagonals $\overline{WY}$ and $\overline{XZ}$ are congruent and are perpendicular bisectors of each other.

5. An isosceles trapezoid has sides of lengths 5, 8, 5, and 14. Find the length of the median.

6. M is the midpoint of hypotenuse $\overline{AB}$. Find AM and $m\angle ACM$.



7. Given: $\angle 1 \cong \angle 2 \cong \angle 3 \cong \angle 4$
Prove: $EFGH$ is a rhombus.

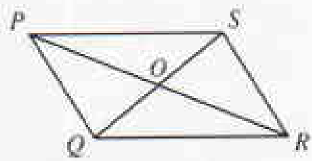


8. $PQRS$ is a ▱.

a. If X is the midpoint of $\overline{PQ}$ and Y is the midpoint of $\overline{SR}$, what special kind of quadrilateral is $XQRY$?

b. Prove your answer to part (a).

c. Draw a line through O intersecting $\overline{PQ}$ at J and $\overline{SR}$ at K. If J and K are not midpoints, what special kind of quadrilateral is $JQRK$?



Application

Rhombuses

Many objects that need to change in size or shape are built in the shape of a rhombus. What makes this shape so useful is that if you keep the lengths of the sides the same, opposite sides remain parallel as you change the measures of the angles. A rhombus also has the property that as you change the measures of the angles, the vertices slide along the lines that contain the diagonals and the diagonals remain perpendicular. Two applications of this property are illustrated in the photographs below.

A rhombic shape can be used to support weight when the height of the object changes but the load must remain balanced. To use the jack shown above, you turn a crank. This brings the two hinges on the horizontal diagonal closer together and forces the hinges on the vertical diagonal farther apart, which lifts the car. Could a jack be in the shape of a parallelogram that is not a rhombus? Would a jack in the shape of a kite work?

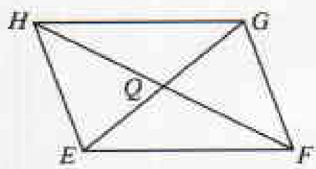
The folding elevator gate shown at the right changes in width, but the vertical bars remain vertical. Some types of fireplace tongs can extend and retract in a similar way. Electric trains sometimes use rhombic arrangements to maintain contact with overhead wires even when the distance between the top of the coach and the wire changes. Can you think of other objects that use this property?

Chapter Summary

1. A parallelogram has these properties:
   a. Opposite sides are parallel.
   b. Opposite sides are congruent.
   c. Opposite angles are congruent.
   d. Diagonals bisect each other.
2. The chart on page 172 lists five ways to prove that a quadrilateral is a parallelogram.
3. If two lines are parallel, then all points on one line are equidistant from the other line.
4. If three parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.
5. A line that contains the midpoint of one side of a triangle and is parallel to another side bisects the third side.
6. The segment that joins the midpoints of two sides of a triangle is parallel to the third side and has a length equal to half the length of the third side.
7. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.
8. Rectangles, rhombuses, and squares are parallelograms with additional properties. Trapezoids and kites are not parallelograms, but are special quadrilaterals with additional properties.
9. The median of a trapezoid is parallel to the bases and has a length equal to half the sum of the lengths of the bases.

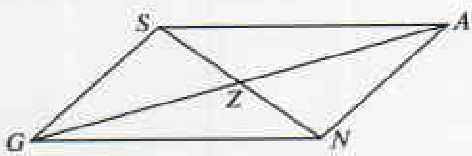
Chapter Review

In parallelogram *EFGH*, $m\angle EFG = 70$. 5–1



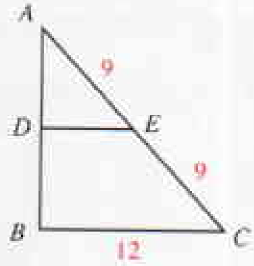
1. $m\angle HEF = \underline{\ ?\ }$
2. If $m\angle EFH = 32$, then $m\angle EHF = \underline{\ ?\ }$.
3. If $HQ = 14$, then $HF = \underline{\ ?\ }$.
4. If $EH = 8x - 7$ and $FG = 5x + 11$, then $x = \underline{\ ?\ }$.

In each exercise you could prove that quad. *SANG* is a parallelogram if one more fact, in addition to those stated, were given. State that fact. 5–2



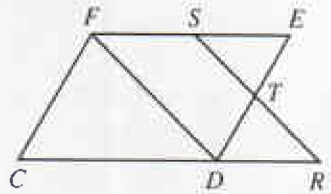
5. $GN = 9$; $NA = 5$; $SA = 9$
6. $\angle ASG \cong \angle GNA$
7. $\overline{SZ} \cong \overline{NZ}$
8. $\overline{SA} \parallel \overline{GN}$; $SA = 17$

State the principal theorem that justifies the statement about the diagram.



9. If $\overline{DE} \parallel \overline{BC}$, then D is the midpoint of $\overline{AB}$. 5–3
10. If D is the midpoint of $\overline{AB}$, then $\overline{DE} \parallel \overline{BC}$.
11. If D is the midpoint of $\overline{AB}$, then $DE = 6$.

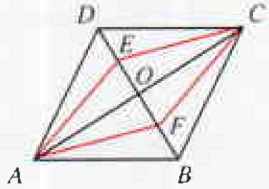
12. Given: ▱$CDEF$; S and T are the midpoints of $\overline{EF}$ and $\overline{ED}$.
Prove: $\overline{SR} \cong \overline{FD}$



Give the most descriptive name for quad. *MNOP*.

13. $\overline{MN} \cong \overline{PO}$; $\overline{MN} \parallel \overline{PO}$ 5–4
14. $\overline{MN} \parallel \overline{PO}$; $\overline{NO} \parallel \overline{MP}$; $\overline{MO} \perp \overline{NP}$
15. $\angle M \cong \angle N \cong \angle O \cong \angle P$
16. $MNOP$ is a rectangle with $MN = NO$.

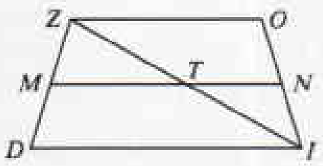
17. Given: $ABCD$ is a rhombus; $DE = BF$
Prove: $AECF$ is a rhombus.



Draw and label a diagram. List, in terms of the diagram, what is given and what is to be proved. Then write a proof.

18. $\overline{PX}$ and $\overline{QY}$ are altitudes of acute $\triangle PQR$, and Z is the midpoint of $\overline{PQ}$. Prove that $\triangle XYZ$ is isosceles.

$\overline{MN}$ is the median of trapezoid *ZOID*.



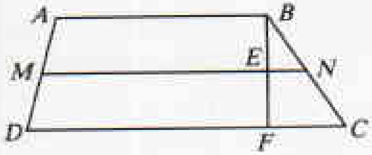
19. The bases of trap. $ZOID$ are _?_ and _?_. 5–5
20. If $ZO = 8$ and $MN = 11$, then $DI =$ _?_.
21. If $ZO = 8$, then $TN =$ _?_.
22. If trap. $ZOID$ is isosceles and $m\angle D = 80$, then $m\angle O =$ _?_.

Chapter Test

Complete each statement with the word *always*, *sometimes*, or *never*.

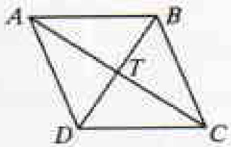
1. A square is _?_ a rectangle.
2. A rectangle is _?_ a rhombus.
3. A rhombus is _?_ a square.
4. A rhombus is _?_ a parallelogram.
5. A trapezoid _?_ has three congruent sides.
6. The diagonals of a trapezoid _?_ bisect each other.
7. The diagonals of a rectangle are _?_ congruent.
8. The diagonals of a parallelogram _?_ bisect the angles.

Trapezoid *ABCD* has median $\overline{MN}$.



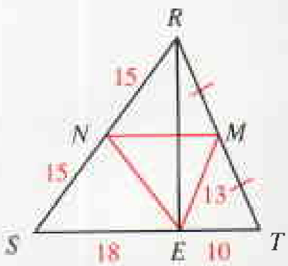
9. If $DC = 42$ and $MN = 35$, then $AB =$ _?_.
10. If $FC = 9$, then $EN =$ _?_.
11. If $AB = 5j + 7k$ and $DC = 9j - 3k$, then $MN =$ _?_.

Can you deduce from the given information that quad. *ABCD* is a parallelogram? If so, what theorem can you use?

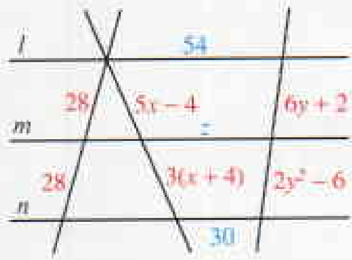


12. $\angle ADC \cong \angle CBA$ and $\angle BAD \cong \angle DCB$
13. $\overline{AD} \parallel \overline{BC}$ and $\overline{AD} \cong \overline{BC}$
14. $AT = CT$ and $DT = \frac{1}{2}DB$
15. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are all congruent.

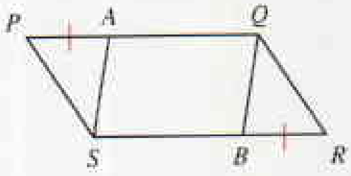
16. $\overline{RE}$ is an altitude of $\triangle RST$.
Find MN, NE, and RT.



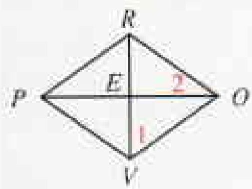
17. $l \parallel m \parallel n$
Find the values of x, y, and z.



18. Given: ▱$PQRS$; $PA = RB$
Prove: $AS = BQ$



19. Given: $\overline{PR} \parallel \overline{VO}$; $\overline{RO} \parallel \overline{PV}$; $\overline{PR} \cong \overline{RO}$
Prove: $\angle 1$ and $\angle 2$ are complementary.



Cumulative Review: Chapters 1–5

A

1. Given two parallel lines n and k, how many planes contain n and k?

2. a. Is it possible for two lines to be neither intersecting nor parallel? If so, what are the lines called?

b. Repeat part (a), replacing *lines* with *planes*.

3. Write the converse of the statement: If you are a member of the skiing club, then you enjoy winter weather.

4. On a number line, point A has a coordinate -5 and B has a coordinate 3. Find the coordinate of the midpoint of $\overline{AB}$.

5. Name the property that justifies the statement: If $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, then $\angle 1 \cong \angle 3$.

In Exercises 6–10, complete each statement about the diagram. Then state the definition, postulate, or theorem that justifies your answer.

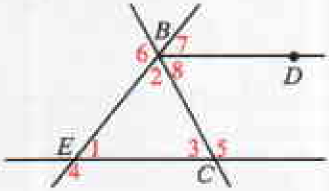
6. $m\angle 1 + m\angle 2 + m\angle 3 =$ ___?___

7. $m\angle 1 + m\angle 4 =$ ___?___

8. $m\angle 1 + m\angle 2 = m\angle$ ___?___

9. If $\overleftrightarrow{EC} \parallel \overleftrightarrow{BD}$, then $\angle 7 \cong$ ___?___.

10. If $\angle 2 \cong \angle 3$, then $\overline{EC} \cong$ ___?___.



Complete each statement.

11. The median to the base of an isosceles triangle ___?___ the vertex angle and is ___?___ to the base.

12. a. If a point lies on the perpendicular bisector of $\overline{AB}$, then the point is equidistant from ___?___.

b. If a point lies on the bisector of $\angle RST$, then the point is equidistant from ___?___.

13. Suppose $\triangle ART \cong \triangle DEB$.

a. $\triangle EBD \cong$ ___?___ **b.** $\overline{AT} \cong$ ___?___ **c.** $m\angle R =$ ___?___

14. If a regular polygon has 40 sides, the measure of each interior angle is ___?___.

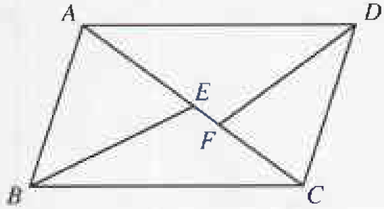
15. When two parallel lines are cut by a transversal, a pair of corresponding angles have measures $2x + 50$ and $3x$. The measures of the angles are ___?___ and ___?___.

B

16. In $\triangle SUN$, $\angle S \cong \angle N$. Given that $SU = 2x + 7$, $UN = 4x - 1$, and $SN = 3x + 4$, find the numerical length of each side.

17. M and N are the midpoints of the legs of trapezoid $EFGH$. If bases $\overline{EF}$ and $\overline{HG}$ have lengths $2r + s$ and $4r - 3s$, express the length of $\overline{MN}$ in terms of r and s.

Given: $ABCD$ is a parallelogram;
$\overline{AD} \cong \overline{AC}$; $\overline{AE} \cong \overline{EC}$
$\angle ADF \cong \angle CDF$; $m\angle DAC = 36$
Complete each statement about the diagram.



18. $\overline{AE} \cong \overline{EC}$, so $\overline{BE}$ is a(n) __?__ of $\triangle ABC$.

19. $\angle ADF \cong \angle CDF$, so $\overline{DF}$ is a(n) __?__ of $\angle ADC$.

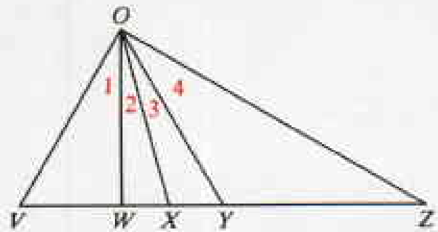
20. $\triangle ADC$ is a(n) __?__ triangle.

21. $m\angle DAC = 36$, so $m\angle ADC =$ __?__ and $m\angle ADF =$ __?__.

22. $\triangle ADF$ is a(n) __?__ triangle.

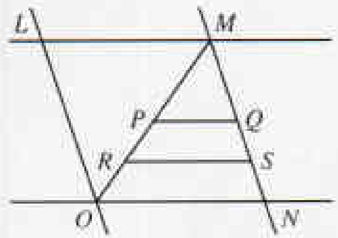
23. $\angle ADC \cong \angle$ __?__ $\cong \angle$ __?__ $\cong \angle$ __?__ $\cong \angle$ __?__.

In the diagram, $m\angle VOZ = 90$.
$\overline{OW}$ is an altitude of $\triangle VOZ$.
$\overline{OX}$ bisects $\angle VOZ$.
$\overline{OY}$ is a median of $\triangle VOZ$.
Find the measures of the four numbered angles.



24. $m\angle Z = 30$ **25.** $m\angle Z = k$

In Exercises 26–29, complete each statement about the diagram. Then state the definition, postulate, or theorem that justifies your answer.



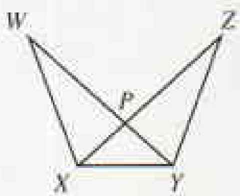
26. If $LM = ON$ and $LO = MN$, then $LMNO$ is a __?__.

27. If $LMNO$ is a rhombus, then $\angle LOM \cong$ __?__ $\cong$ __?__ $\cong$ __?__.

28. If $MP = PO$ and $\overline{PQ} \parallel \overline{ON}$, then Q is the __?__ of __?__.

29. If $\overline{PQ} \parallel \overline{ON}$, $PR = RO$ and $QS = SN$, then $RS = \frac{1}{2}($__?__ + __?__$)$.

30. Given: $WP = ZP$; $PY = PX$
Prove: $\angle WXY \cong \angle ZYX$



31. Given: $\overline{AD} \cong \overline{BC}$; $\overline{AD} \parallel \overline{BC}$
Prove: $\overline{EF} \cong \overline{FG}$