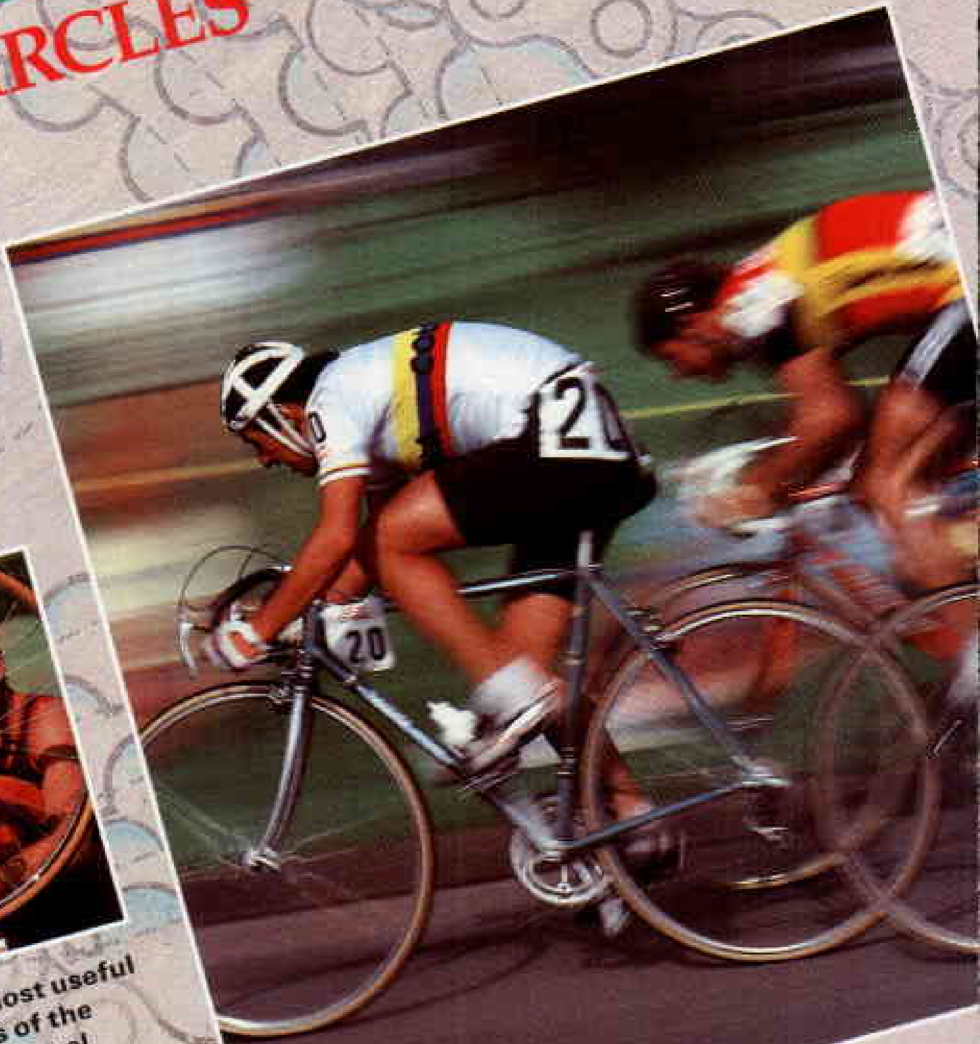
9 CIRCLES



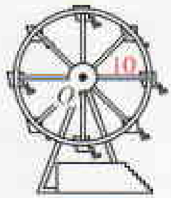
One of the most useful applications of the circle is the wheel. From ancient times until today, the wheel has made an enormous contribution to progress in travel, transportation, industry, and other elements of civilization.

Tangents, Arcs, and Chords

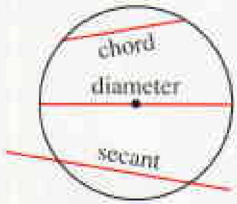
Objectives

1. Define a circle, a sphere, and terms related to them.
2. Recognize circumscribed and inscribed polygons and circles.
3. Apply theorems that relate tangents and radii.
4. Define and apply properties of arcs and central angles.
5. Apply theorems about the chords of a circle.

9-1 *Basic Terms*

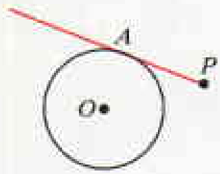


A **circle** is the set of points in a plane at a given distance from a given point in that plane. The given point is the **center** of the circle and the given *distance* is *the* **radius.** Any *segment* that joins the center to a point of the circle is called *a* radius. All radii of a circle are congruent. The rim of the Ferris wheel shown is a circle with center O ($\odot O$) and radius 10.



A **chord** is a segment whose endpoints lie on a circle. A **secant** is a line that contains a chord. A **diameter** is a chord that contains the center of a circle. (Like the word *radius*, the word *diameter* can refer to *the* length of a segment or to *a* segment.)

A **tangent** is a line in the plane of a circle that intersects the circle in exactly one point, called the **point of tangency.** The *tangent ray* $\overrightarrow{PA}$ and *tangent segment* $\overline{PA}$ are often called tangents.



$\overleftrightarrow{AP}$ is tangent to $\odot O$.

$\odot O$ is tangent to $\overleftrightarrow{AP}$.

A is the point of tangency.

A **sphere** with center O and radius r is the set of all points in space at a distance r from point O. Many of the terms used with spheres are the same as those used with circles.

$\overline{OA}$, $\overline{OB}$, and $\overline{OD}$ are radii.

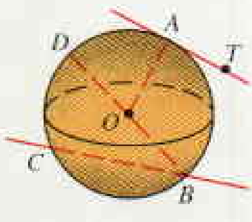
$\overline{BD}$ is a diameter.

$\overline{BC}$ is a chord.

$\overleftrightarrow{BC}$ is a secant.

$\overleftrightarrow{AT}$ is a tangent.

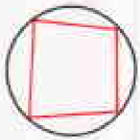
$\overline{AT}$ is a tangent segment.



Congruent circles (or **spheres**) are circles (or spheres) that have congruent radii.

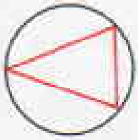
Concentric circles are circles that lie in the same plane and have the same center. The rings of the target illustrate concentric circles. **Concentric spheres** are spheres that have the same center.

A polygon is **inscribed in a circle** and the circle is **circumscribed about the polygon** when each vertex of the polygon lies on the circle.

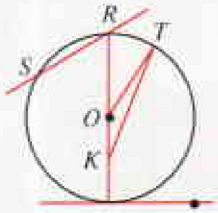


Inscribed polygons

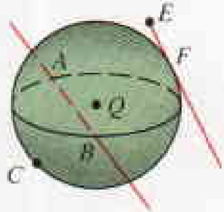
Circumscribed circles



Classroom Exercises



1. Name three radii of $\odot O$.
2. Name a diameter.
3. Consider $\overline{RS}$ and $\overleftrightarrow{RS}$. Which is a chord and which is a secant?
4. Why is $\overline{TK}$ not a chord?
5. Name a tangent to $\odot O$.
6. What name is given to point L?



7. Name a line tangent to sphere Q.
8. Name a secant of the sphere and a chord of the sphere.
9. Name 4 radii. (None are drawn in the diagram.)
10. What is the diameter of a circle with radius 8? 5.2? $4\sqrt{3}$? j?
11. What is the radius of a sphere with diameter 14? 13? 5.6? $6n$?

Written Exercises

A 1. Draw a circle and several parallel chords. What do you think is true of the midpoints of all such chords?

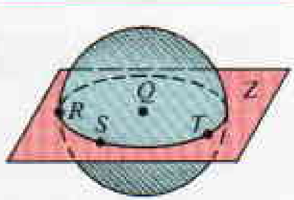
2. Draw a circle with center O and a line $\overleftrightarrow{TS}$ tangent to $\odot O$ at T. Draw $\overline{OT}$, and use a protractor to find $m\angle OTS$.

3. **a.** Draw a right triangle inscribed in a circle.
   b. What do you know about the midpoint of the hypotenuse?
   c. Where is the center of the circle?
   d. If the legs of the right triangle are 6 and 8, find the radius of the circle.

4. Plane Z passes through the center of sphere Q.

a. Explain why $QR = QS = QT$.

b. Explain why the intersection of the plane and the sphere is a circle. (The intersection of a sphere with any plane passing through the center of the sphere is called a **great circle** of the sphere.)



5. The radii of two concentric circles are 15 cm and 7 cm. A diameter $\overline{AB}$ of the larger circle intersects the smaller circle at C and D. Find two possible values for AC.

For each exercise draw a circle and inscribe the polygon in the circle.

6. A rectangle

7. A trapezoid

8. An obtuse triangle

9. A parallelogram

10. An acute isosceles triangle

11. A quadrilateral $PQRS$, with $\overline{PR}$ a diameter

For each exercise draw $\odot O$ with radius 12. Then draw radii $\overline{OA}$ and $\overline{OB}$ to form an angle with the measure named. Find the length of $\overline{AB}$.

B **12.** $m\angle AOB = 90$

13. $m\angle AOB = 180$

14. $m\angle AOB = 60$

15. $m\angle AOB = 120$

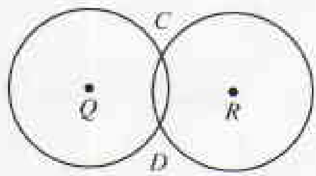
16. Draw two points A and B and several circles that pass through A and B. Locate the centers of these circles. On the basis of your experiment, complete the following statement:
The centers of all circles passing through A and B lie on __?__.
Write an argument to support your statement.

17. $\odot Q$ and $\odot R$ are congruent circles that intersect at C and D. $\overline{CD}$ is called the *common chord* of the circles.

a. What kind of quadrilateral is $QDRC$? Why?

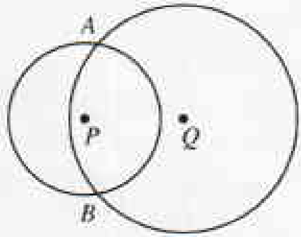
b. $\overline{CD}$ must be the perpendicular bisector of $\overline{QR}$. Why?

c. If $QC = 17$ and $QR = 30$, find CD.



18. Draw two congruent circles with radii 6 each passing through the center of the other. Find the length of their common chord.

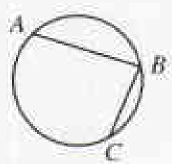
C **19.** $\odot P$ and $\odot Q$ have radii 5 and 7 and $PQ = 6$. Find the length of the common chord $\overline{AB}$. (*Hint*: $APBQ$ is a kite and $\overline{PQ}$ is the perpendicular bisector of $\overline{AB}$. See Exercise 28, page 193. Let N be the intersection of $\overline{PQ}$ and $\overline{AB}$, and let $PN = x$ and $AN = y$. Write two equations in terms of x and y.)



20. Draw a diagram similar to the one shown, but much larger. Carefully draw the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$.

a. The perpendicular bisectors intersect in a point. Where does that point appear to be?

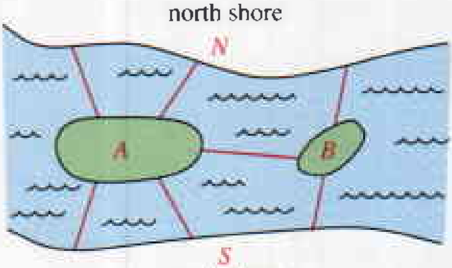
b. Write an argument that justifies your answer to part (a).

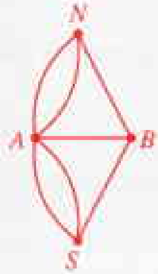


Extra

Networks

The Pregel River flows through the old city of Koenigsberg, now Kaliningrad. Once, seven bridges joined the shores and the two islands in the river as shown in the diagram at the left below. A popular problem of that time was to try to walk across all seven bridges without crossing any bridge more than once. Can you find a way to do it?



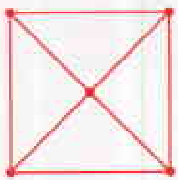


Mathematician Leonard Euler analyzed this problem using a diagram called a *network*, shown at the right above. He represented each land mass by a point (called a vertex) and each bridge by an arc. He then classified each vertex with an odd number of arcs coming from it as *odd* and each vertex with an even number of arcs as *even*. From here Euler discovered which networks can be traced without backtracking, that is, without drawing over an arc twice.

Exercises

Find the number of odd and even vertices in each network. Imagine traveling each network to see if it can be traced without backtracking.

1.



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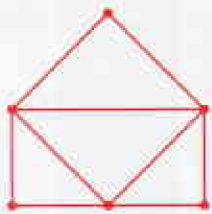

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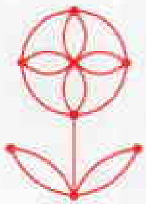


The number of odd vertices will tell you whether or not a network can be traced without backtracking. Do you see how? If not, read on.

4. Suppose that a given network can be traced without backtracking.
   a. Consider a vertex that is neither the start nor end of a journey through this network. Is such a vertex odd or even?
   b. Now consider the two vertices at the start and finish of a journey through this network. Can both of these vertices be odd? even?
   c. Can just one of the start and finish vertices be odd?

5. Tell why it is impossible to walk across the seven bridges of Koenigsberg without crossing any bridge more than once.

9-2 *Tangents*

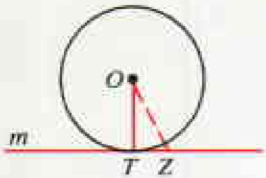
In Written Exercise 2 on page 330 you had the chance to preview the next theorem about tangents and radii.

Theorem 9-1

If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.

Given: m is tangent to $\odot O$ at T.

Prove: $\overline{OT} \perp m$



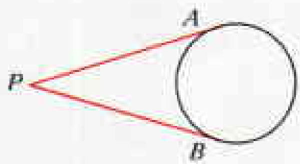
Proof:

Assume temporarily that $\overline{OT}$ is not perpendicular to m. Then the perpendicular segment from O to m intersects m in some other point Z. Draw $\overline{OZ}$. By Corollary 1, page 220, the perpendicular segment from O to m is the shortest segment from O to m, so $OZ < OT$. Because tangent m intersects $\odot O$ only in point T, Z lies outside $\odot O$, and $OZ > OT$. The statements $OZ < OT$ and $OZ > OT$ are contradictory. Thus the temporary assumption must be false. It follows that $\overline{OT} \perp m$.

Corollary

Tangents to a circle from a point are congruent.

In the figure, $\overline{PA}$ and $\overline{PB}$ are tangent to the circle at A and B. By the corollary, $\overline{PA} \cong \overline{PB}$. For a proof, see Classroom Exercise 4.



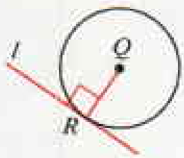
Theorem 9-2 is the converse of Theorem 9-1. Its proof is left as Exercise 22.

Theorem 9-2

If a line in the plane of a circle is perpendicular to a radius at its outer endpoint, then the line is tangent to the circle.

Given: Line l in the plane of $\odot Q$;
$l \perp$ radius $\overline{QR}$ at R

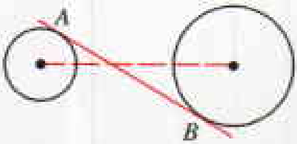
Prove: l is tangent to $\odot Q$.



When each side of a polygon is tangent to a circle, the polygon is said to be **circumscribed about the circle** and the circle is **inscribed in the polygon.**

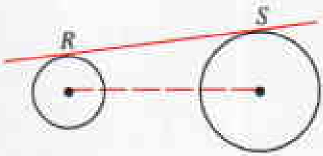
A line that is tangent to each of two coplanar circles is called a **common tangent.**

A common *internal* tangent intersects the segment joining the centers.



$\overleftrightarrow{AB}$ is a common internal tangent. Can you find another one that has not been drawn?

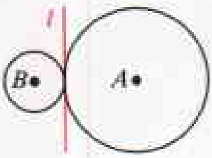
A common *external* tangent does *not* intersect the segment joining the centers.



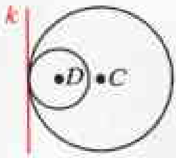
$\overleftrightarrow{RS}$ is a common external tangent. Can you find another one that has not been drawn?

A circle can be tangent to a line, but it can also be tangent to another circle. **Tangent circles** are coplanar circles that are tangent to the same line at the same point.

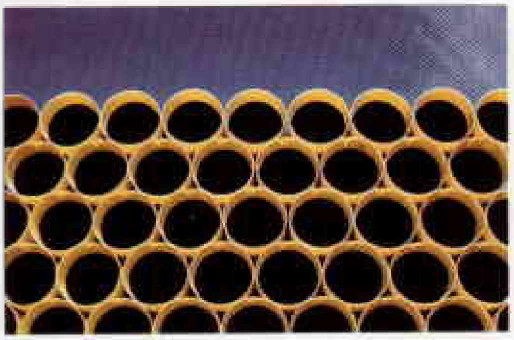
$\odot A$ and $\odot B$ are *externally* tangent.



$\odot C$ and $\odot D$ are *internally* tangent.

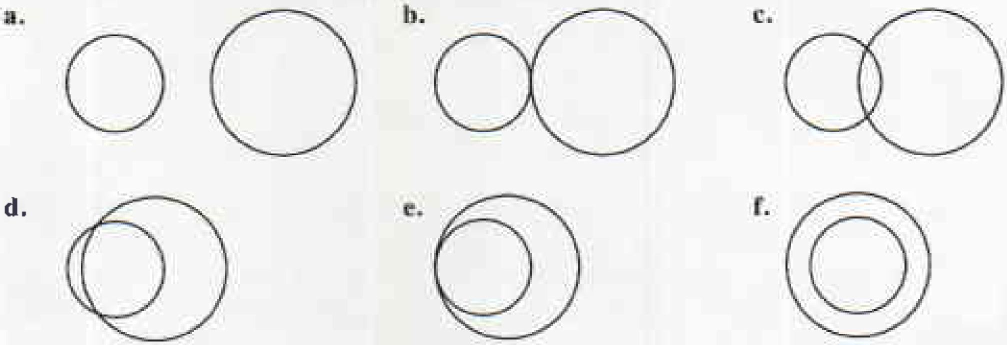


The ends of the plastic industrial pipes shown in the photograph illustrate externally tangent circles. Notice that when a circle is surrounded by tangent circles of the same radius, six of these circles fit exactly around the inner circle.



Classroom Exercises

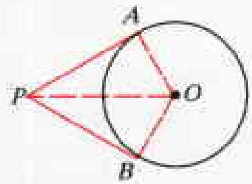
1. How many common external tangents can be drawn to the two circles?



2. How many common internal tangents can be drawn to each pair of circles in Exercise 1 above?

3. **a.** Which pair of circles shown above are externally tangent?
b. Which pair are internally tangent?

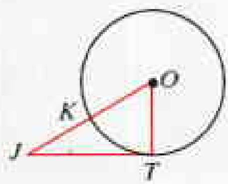
4. Given: $\overline{PA}$ and $\overline{PB}$ are tangents to $\odot O$.
Use the diagram at the right to explain how the corollary on page 333 follows from Theorem 9-1.



5. In the diagram, which pairs of angles are congruent? Which pairs of angles are complementary? Which pairs of angles are supplementary?

Written Exercises

$\overline{JT}$ is tangent to $\odot O$ at T. Complete.



A

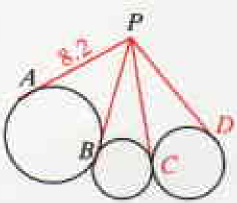
1. If $OT = 6$ and $JO = 10$, then JT = ____?____.

2. If $OT = 6$ and $JT = 10$, then JO = ____?____.

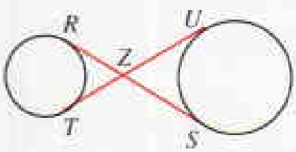
3. If $m\angle TOJ = 60$ and $OT = 6$, then JO = ____?____.

4. If $JK = 9$ and $KO = 8$, then JT = ____?____.

5. The diagram below shows tangent lines and circles. Find PD.

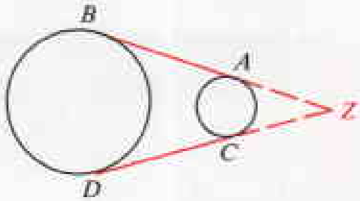


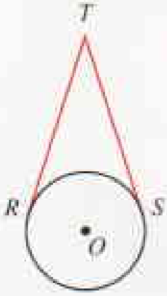
6. $\overline{RS}$ and $\overline{TU}$ are common internal tangents to the circles. If $RZ = 4.7$ and $ZU = 7.3$, find RS and TU.



7. a. What do you think is true of common external tangents $\overline{AB}$ and $\overline{CD}$? Prove it.

b. Will your results in part (a) be true if the circles are congruent?





8. Given: $\overline{TR}$ and $\overline{TS}$ are tangents to $\odot O$ from T;
$m\angle RTS = 36$

a. Copy the diagram. Draw $\overline{RS}$ and find $m\angle TSR$ and $m\angle TRS$.

b. Draw radii $\overline{OS}$ and $\overline{OR}$ and find $m\angle ORS$ and $m\angle OSR$.

c. Find $m\angle ROS$.

d. Does your result in part (c) support one of your conclusions about angles in Classroom Exercise 5? Explain.

9. Draw $\odot O$ with perpendicular radii $\overline{OX}$ and $\overline{OY}$. Draw tangents to the circle at X and Y.

a. If the tangents meet at Z, what kind of figure is $OXZY$? Explain.

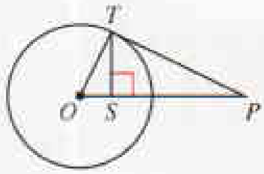
b. If $OX = 5$, find OZ.

10. Given: $\overline{PT}$ is tangent to $\odot O$ at T; $\overline{TS} \perp \overline{PO}$
Complete the following statements.

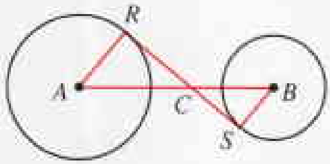
a. TS is the geometric mean between _?_ and _?_.

b. TO is the geometric mean between _?_ and _?_.

c. If $OS = 6$ and $SP = 24$, $TS =$ _?_ and $TP =$ _?_.



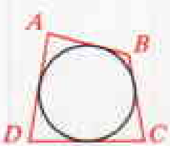
11. Given: $\overline{RS}$ is a common internal tangent to $\odot A$ and $\odot B$.
Explain why $\frac{AC}{BC} = \frac{RC}{SC}$.



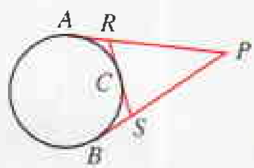
B **12.** Discover and prove a theorem about two lines tangent to a circle at the endpoints of a diameter.

13. Is there a theorem about spheres related to the theorem in Exercise 12? If so, state the theorem.

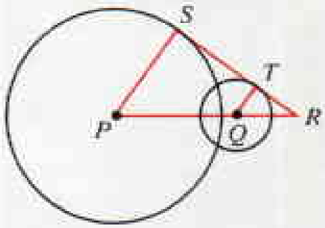
14. Quad. $ABCD$ is circumscribed about a circle. Discover and prove a relationship between $AB + DC$ and $AD + BC$.



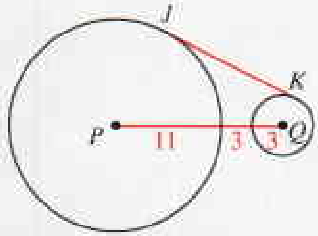
15. $\overline{PA}$, $\overline{PB}$, and $\overline{RS}$ are tangents. Explain why $PR + RS + SP = PA + PB$.



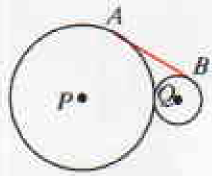
16. $\overline{SR}$ is tangent to $\odot P$ and $\odot Q$. $QT = 6$; $TR = 8$; $PR = 30$. $PQ = \underline{\ ?\ }$; $PS = \underline{\ ?\ }$; $ST = \underline{\ ?\ }$.



17. $\overline{JK}$ is tangent to $\odot P$ and $\odot Q$. $JK = \underline{\ ?\ }$ (*Hint*: What kind of quadrilateral is $JPQK$?)



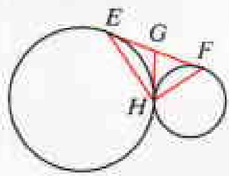
18. Circles P and Q have radii 6 and 2 and are tangent to each other. Find the length of their common external tangent $\overline{AB}$. (*Hint*: Draw $\overline{PQ}$, $\overline{PA}$, and $\overline{QB}$.)



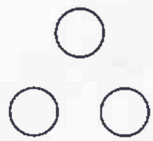
19. Given: Two tangent circles; $\overline{EF}$ is a common external tangent; $\overline{GH}$ is the common internal tangent.

a. Discover and prove something interesting about point G.

b. Discover and prove something interesting about $\angle EHF$.



20. Three circles are shown. How many circles tangent to all three of the given circles can be drawn?



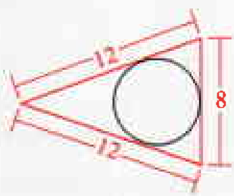
21. Suppose the three circles represent three spheres.

a. How many planes tangent to each of the spheres can be drawn?

b. How many spheres tangent to all three spheres can be drawn?

22. Prove Theorem 9-2. (*Hint*: Write an indirect proof.)

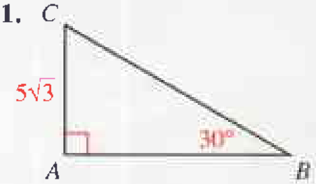
23. Find the radius of the circle inscribed in the triangle.



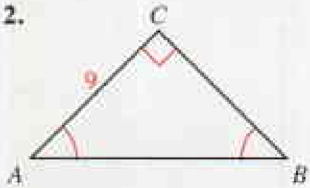
Mixed Review Exercises

Find AB. In Exercise 3, $\overline{CB}$ is tangent to $\odot A$.

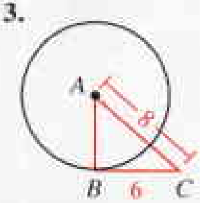
1.



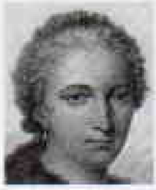
2.



3.

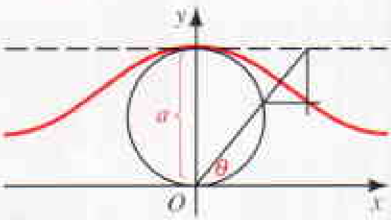


Biographical Note — *Maria Gaetana Agnesi*



Maria Gaetana Agnesi (1718–1799) was born in Milan, Italy. A child prodigy, she had mastered seven languages by the age of thirteen. Between the ages of twenty and thirty she compiled the works of the mathematicians of her time into two volumes on calculus, called *Analytical Institutions*. This was an enormous task, since the mathematicians had originally published their results in different languages and had used a variety of methods of approach.

Agnesi's volumes were praised as clear, methodical, and comprehensive. They were translated into English and French and were widely used as textbooks. One of the most famous aspects of Agnesi's volumes was an exercise in analytic geometry and the discussion of a curve called a *versirea*, shown at the left below. The name, derived from the Latin *vertere*, "to turn," was apparently mistranslated into English texts as "witch." Thus the curve is commonly known as the "witch of Agnesi."



Due to Agnesi's scholarship, she was elected to the Bologna Academy of Sciences and in 1750 she was appointed honorary professor in mathematics at the University of Bologna, shown at the left.

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

Draw parallelogram *ABCD*. Draw four circles as follows.

(1) Use *A*, *B*, and *D* to draw circle *E*.
(2) Use *A*, *D*, and *C* to draw circle *F*.
(3) Use *B*, *C*, and *D* to draw circle *G*.
(4) Use *A*, *B*, and *C* to draw circle *H*.

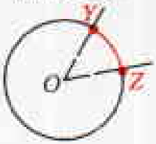
Connect the centers of the circles to get quad. *EFGH*.
Compare quad. *ABCD* with quad. *EFGH*. What do you notice?

Repeat on other types of quadrilaterals: a rhombus, a trapezoid, a rectangle, and an isosceles trapezoid. What do you notice?

9-3 *Arcs and Central Angles*

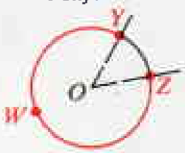
A **central angle** of a circle is an angle with its vertex at the center of the circle. In the diagrams below, $\angle YOZ$ is a central angle. An *arc* is an unbroken part of a circle. Two points Y and Z on a circle O are always the endpoints of two arcs. Y and Z and the points of $\odot O$ in the interior of $\angle YOZ$ form a **minor arc.** Y and Z and the remaining points of $\odot O$ form a **major arc.** If Y and Z are the endpoints of a diameter, then the two arcs are called **semicircles.** A minor arc is named by its endpoints: $\overset{\frown}{YZ}$ is read "arc YZ." You use three letters to name a semicircle or a major arc: $\overset{\frown}{YWZ}$ is read "arc YWZ."

Minor Arc



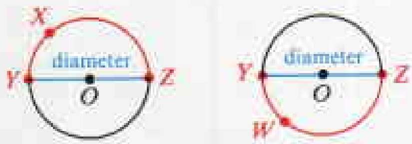
$\overset{\frown}{YZ}$

Major Arc



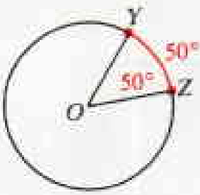
$\overset{\frown}{YWZ}$

Semicircles



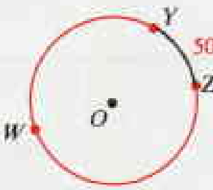
$\overset{\frown}{YXZ}$ $\overset{\frown}{YWZ}$

The **measure of a minor arc** is defined to be the measure of its central angle. In the diagram at the left below, $m\overset{\frown}{YZ}$ represents the measure of minor arc YZ. In the middle diagram, can you see why the **measure of a major arc** is 360 minus the measure of its minor arc? The third diagram shows that the **measure of a semicircle** is 180.



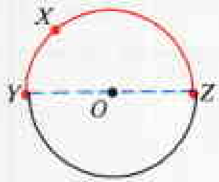
Minor Arc

$m\overset{\frown}{YZ} = m\angle YOZ$
$= 50$



Major Arc

$m\overset{\frown}{YWZ} = 360 - m\overset{\frown}{YZ}$
$= 360 - 50 = 310$



Semicircle

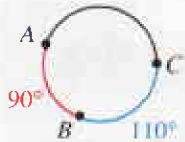
$m\overset{\frown}{YXZ} = 180$

Adjacent arcs of a circle are arcs that have exactly one point in common. The following postulate can be used to find the measure of an arc formed by two adjacent arcs.

Postulate 16 *Arc Addition Postulate*

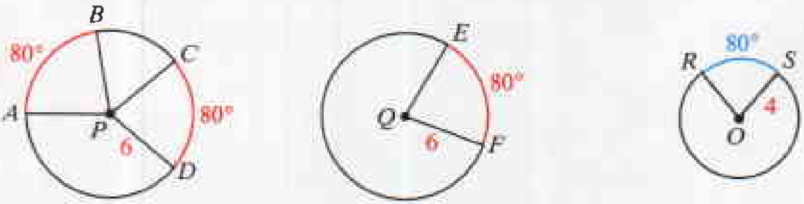
The measure of the arc formed by two adjacent arcs is the sum of the measures of these two arcs.

Applying the Arc Addition Postulate to the circle shown at the right, we have



$$m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{ABC}$$
$$90 + 110 = 200$$

Congruent arcs are arcs, in the same circle or in congruent circles, that have equal measures. In the diagram below, $\odot P$ and $\odot Q$ are congruent circles and $\overset{\frown}{AB} \cong \overset{\frown}{CD} \cong \overset{\frown}{EF}$. However, $\overset{\frown}{EF}$ is not congruent to $\overset{\frown}{RS}$ even though both arcs have the same degree measure, because $\odot Q$ is not congruent to $\odot O$.



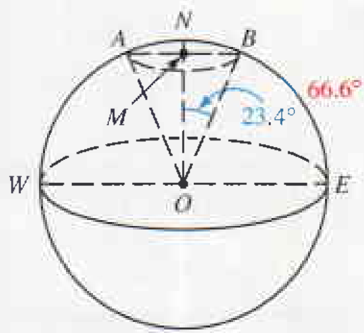
Notice that each of the congruent arcs above has an 80° central angle, so these congruent arcs have congruent central angles. The relationship between congruence of minor arcs and congruence of their central angles is stated in Theorem 9-3 below. This theorem follows immediately from the definition of congruent arcs.

Theorem 9-3

In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent.

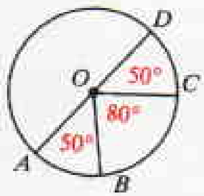
Example The radius of the Earth is about 6400 km. The latitude of the Arctic Circle is 66.6° North. (That is, in the figure, $m\overset{\frown}{BE} = 66.6$.) Find the radius of the Arctic Circle.

Solution Let N be the North Pole and let $\overline{ON}$ intersect $\overline{AB}$ in M. Since $m\overset{\frown}{NE} = 90$, $m\overset{\frown}{NB} = 90 - 66.6 = 23.4$ and $m\angle NOB = 23.4$. Similarly, $m\angle NOA = 23.4$. Since $\triangle AOB$ is isosceles and $\overline{OM}$ bisects the vertex $\angle AOB$, (1) M is the midpoint of $\overline{AB}$ (and thus the center of the Arctic Circle) and (2) $\overline{OM} \perp \overline{AB}$. Using trigonometry in right $\triangle MOB$:



$$\sin 23.4° = \frac{MB}{OB}$$
$$MB = OB \cdot \sin 23.4°$$
$$MB \approx 6400(0.3971)$$
$$MB \approx 2500 \text{ km}$$

Classroom Exercises

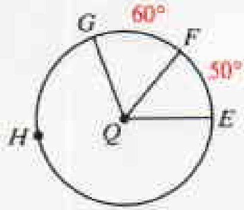


1. Using the letters shown in the diagram, name:
   - **a.** two central angles
   - **b.** a semicircle
   - **c.** two minor arcs
   - **d.** two major arcs

In Exercises 2–7 find the measure of the arc.

2. $\overset{\frown}{AB}$ **3.** $\overset{\frown}{AC}$ **4.** $\overset{\frown}{ABD}$

5. $\overset{\frown}{BAD}$ **6.** $\overset{\frown}{CDA}$ **7.** $\overset{\frown}{CDB}$



In Exercises 8–13 find the measure of the angle or the arc named.

8. $\angle GQF$ **9.** $\angle EQF$ **10.** $\angle GQE$

11. $\overset{\frown}{GE}$ **12.** $\overset{\frown}{GHE}$ **13.** $\overset{\frown}{EHF}$

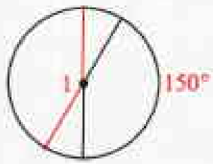
Written Exercises

Find the measure of central $\angle 1$.

A **1.**

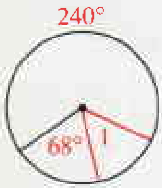
2.

3.

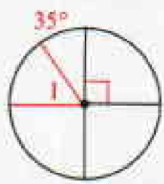


4.

5.



6.



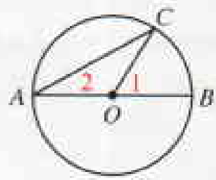
7. At 11 o'clock the hands of a clock form an angle of $\underline{\ ?\ }$°.

8. The hands of a clock form a 120° angle at $\underline{\ ?\ }$ o'clock and at $\underline{\ ?\ }$ o'clock.

9. **a.** Draw a circle. Place points A, B, and C on it in such positions that $m\overset{\frown}{AB} + m\overset{\frown}{BC}$ does not equal $m\overset{\frown}{AC}$.

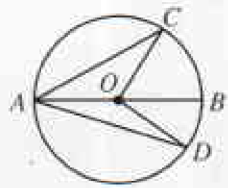
b. Does your example in part (a) contradict Postulate 16?

Complete the tables in Exercises 10 and 11.



10.

$m\overset{\frown}{CB}$	60	70	?	?	?
$m\angle 1$	?	?	56	?	?
$m\angle 2$	?	?	?	25	x



11.

$m\overset{\frown}{CB}$	70	60	66	60	p
$m\overset{\frown}{BD}$	30	28	?	?	q
$m\angle COD$	?	?	100	?	?
$m\angle CAD$	?	?	?	52	?

12. Use a compass to draw a large $\odot O$. Draw a central $\angle AOB$.

a. Label three other points P, Q, and R that are on $\odot O$ but not on $\overset{\frown}{AB}$. Then draw $\angle APB$, $\angle AQB$, and $\angle ARB$.

b. Use a protractor to find $m\angle AOB$, $m\angle APB$, $m\angle AQB$, and $m\angle ARB$.

c. What is the relationship between $m\angle APB$, $m\angle AQB$, and $m\angle ARB$? What is the relationship between $m\angle AOB$ and $m\angle APB$?

13. a. Draw three large circles and inscribe a different-shaped quadrilateral $ABCD$ in each.

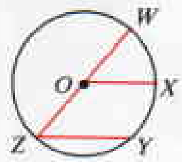
b. Use a protractor to measure all the angles.

c. Compute $m\angle A + m\angle C$ and $m\angle B + m\angle D$.

d. What is the relationship between opposite angles of an inscribed quadrilateral?

B

14. Given: $\overline{WZ}$ is a diameter of $\odot O$; $\overline{OX} \parallel \overline{ZY}$
Prove: $\overset{\frown}{WX} \cong \overset{\frown}{XY}$
(*Hint*: Draw $\overline{OY}$.)



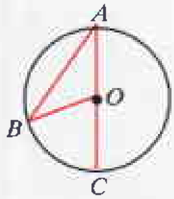
15. Given: $\overline{WZ}$ is a diameter of $\odot O$;
$m\overset{\frown}{WX} = m\overset{\frown}{XY} = n$
Prove: $m\angle Z = n$

16. $\overline{AC}$ is a diameter of $\odot O$.

a. If $m\angle A = 35$, then $m\angle B =$ _?_, $m\angle BOC =$ _?_, and $m\overset{\frown}{BC} =$ _?_.

b. If $m\angle A = n$, then $m\overset{\frown}{BC} =$ _?_.

c. If $m\overset{\frown}{BC} = 6k$, then $m\angle A =$ _?_.



In Exercises 17–20, the latitude of a city is given. Sketch the Earth and a circle of latitude through the city. Find the radius of this circle.

17. Milwaukee, Wisconsin; 43°N

18. Columbus, Ohio; 40°N

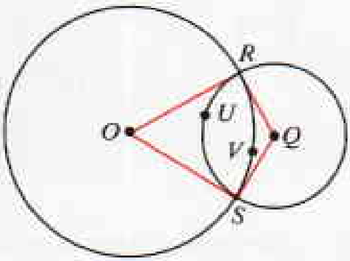
19. Sydney, Australia; 34°S

20. Rio de Janeiro; 23°S

C

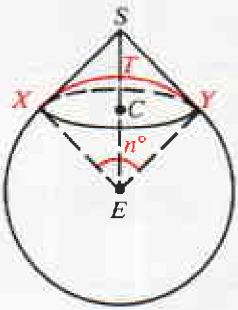
21. Given: $\odot O$ and $\odot Q$ intersect at R and S;
$m\overset{\frown}{RVS} = 60$; $m\overset{\frown}{RUS} = 120$

Prove: $\overline{OR}$ is tangent to $\odot Q$;
$\overline{QR}$ is tangent to $\odot O$.



22. Given: $\overline{AB}$ is a diameter of $\odot Z$; points J and K lie on $\odot Z$ with $m\overset{\frown}{AJ} = m\overset{\frown}{BK}$. Discover and prove something about $\overline{JK}$. (*Hint*: There are two possibilities, depending on whether $\overset{\frown}{AJ}$ and $\overset{\frown}{BK}$ lie on the same side of $\overline{AB}$ or on opposite sides. So your statement will be of the *either* . . . *or* type.)

The diagram, not drawn to scale, shows satellite S above the Earth, represented as sphere E. All lines tangent to the Earth from S touch the Earth at points on a circle with center C. Any two points on the Earth's surface on or above that circle can communicate with each other via S. X and Y are as far apart as two communication points can be. The Earth distance between X and Y equals the length of $\overset{\frown}{XTY}$, which equals $\frac{n}{360}$ · circumference of the Earth. That circumference is approximately 40,200 km and the radius of the Earth is approximately 6400 km.

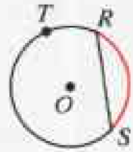


23. The photograph above shows the view from Gemini V looking north over the Gulf of California toward Los Angeles. The orbit of Gemini V ranged from 160 km to 300 km above the Earth. Take S to be 300 km above the Earth. That is, $ST = 300$ km. Find the Earth distance, rounded to the nearest 100 km, between X and Y. (*Hint*: Since you can find a value for $\cos \frac{n^\circ}{2}$ you can determine n°.)

24. Repeat Exercise 23, but with S twice as far from the Earth. Note that the distance between X and Y is not twice as great as before.

9-4 *Arcs and Chords*

In $\odot O$ shown at the right, $\overline{RS}$ cuts off two arcs, $\overset{\frown}{RS}$ and $\overset{\frown}{RTS}$. We speak of $\overset{\frown}{RS}$, the minor arc, as being *the arc of chord* $\overline{RS}$.



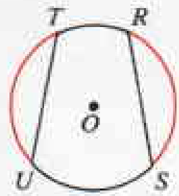
Theorem 9-4

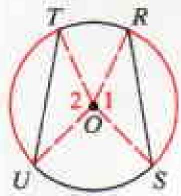
In the same circle or in congruent circles:

(1) Congruent arcs have congruent chords.

(2) Congruent chords have congruent arcs.

Here is a paragraph proof of part (1) for one circle. You will be asked to write a paragraph proof of part (2) in Written Exercise 16.





Given: $\odot O$; $\overset{\frown}{RS} \cong \overset{\frown}{TU}$

Prove: $\overline{RS} \cong \overline{TU}$

Proof:

Draw radii $\overline{OR}$, $\overline{OS}$, $\overline{OT}$, and $\overline{OU}$. $\overline{OR} \cong \overline{OT}$ and $\overline{OS} \cong \overline{OU}$ because they are all radii of the same circle. Since $\overset{\frown}{RS} \cong \overset{\frown}{TU}$, central angles 1 and 2 are congruent. Then $\triangle ROS \cong \triangle TOU$ by SAS and corresponding parts $\overline{RS}$ and $\overline{TU}$ are congruent.

A point Y is called the *midpoint* of $\overset{\frown}{XYZ}$ if $\overset{\frown}{XY} \cong \overset{\frown}{YZ}$. Any line, segment, or ray that contains Y bisects $\overset{\frown}{XYZ}$.

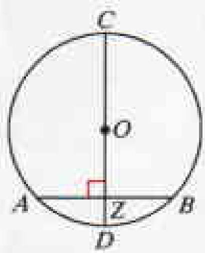
Theorem 9-5

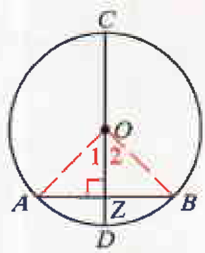
A diameter that is perpendicular to a chord bisects the chord and its arc.

Given: $\odot O$; $\overline{CD} \perp \overline{AB}$

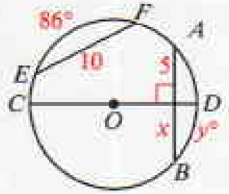
Prove: $\overline{AZ} \cong \overline{BZ}$; $\overset{\frown}{AD} \cong \overset{\frown}{BD}$

Plan for Proof: Draw $\overline{OA}$ and $\overline{OB}$. Then use the HL Theorem to prove that $\triangle OZA \cong \triangle OZB$. Then use corresponding parts of congruent triangles to show that $\overline{AZ} \cong \overline{BZ}$ and $\angle 1 \cong \angle 2$. Finally, apply the theorem that congruent central angles have congruent arcs.





Example 1 Find the values of x and y.



Solution Diameter $\overline{CD}$ bisects chord $\overline{AB}$, so $x = 5$. (Theorem 9-5)

$\overline{AB} \cong \overline{EF}$, so $m\widehat{AB} = 86$. (Theorem 9-4)

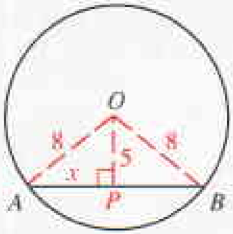
Diameter $\overline{CD}$ bisects $\widehat{AB}$, so $y = 43$. (Theorem 9-5)

Recall (page 154) that the distance from a point to a line is the length of the perpendicular segment from the point to the line. This definition is used in the following example.

Example 2 Find the length of a chord that is a distance 5 from the center of a circle with radius 8.

Solution Draw the perpendicular segment, $\overline{OP}$, from O to $\overline{AB}$.

$$x^2 + 5^2 = 8^2$$
$$x^2 + 25 = 64$$
$$x^2 = 39$$
$$x = \sqrt{39}$$



By Theorem 9-5, $\overline{OP}$ bisects $\overline{AB}$ so
$AB = 2 \cdot AP = 2x = 2\sqrt{39}$.

It should be clear that *all* chords in $\odot O$ above that are a distance 5 from center O will have length $2\sqrt{39}$. Thus, all such chords are congruent, as stated in part (1) of the next theorem. You will prove part (2) of the theorem as Classroom Exercise 6.

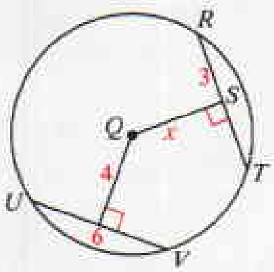
Theorem 9-6

In the same circle or in congruent circles:

(1) Chords equally distant from the center (or centers) are congruent.

(2) Congruent chords are equally distant from the center (or centers).

Example 3 Find the value of x.

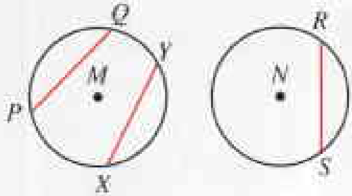


Solution S is the midpoint of $\overline{RT}$, so $RT = 6$. (Theorem 9-5)

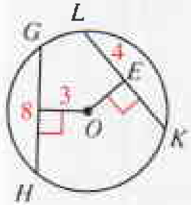
$\overline{RT} \cong \overline{UV}$, so $x = 4$. (Theorem 9-6)

Classroom Exercises

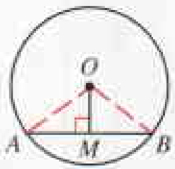
1. If $\overline{PQ} \cong \overline{XY}$, can you conclude that $\overset{\frown}{PQ} \cong \overset{\frown}{XY}$? Why or why not?
2. If $\overline{PQ} \cong \overline{RS}$, can you conclude that $\overset{\frown}{PQ} \cong \overset{\frown}{RS}$? Why or why not?



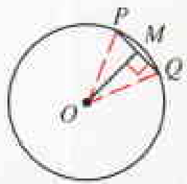
3. Study the diagram at the right and tell what theorem justifies each statement.
   a. $LK = 8$
   b. $OE = 3$
   c. $\overset{\frown}{LK} \cong \overset{\frown}{GH}$



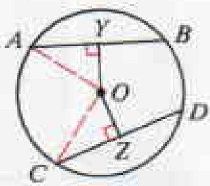
4. $AB = 16$
   $OM = 6$
   radius = ?



5. $PQ = 10$
   radius = 13
   OM = ?



6. Supply reasons to complete a proof of Theorem 9-6, part (2), for one circle.
   Given: $\odot O$; $\overline{AB} \cong \overline{CD}$; $\overline{OY} \perp \overline{AB}$; $\overline{OZ} \perp \overline{CD}$
   Prove: $OY = OZ$



Proof:

Statements	Reasons
1. Draw radii $\overline{OA}$ and $\overline{OC}$.	1. ?
2. $\overline{OY} \perp \overline{AB}$; $\overline{OZ} \perp \overline{CD}$	2. ?
3. $\overline{AB} \cong \overline{CD}$, or $AB = CD$	3. ?
4. $\frac{1}{2}AB = \frac{1}{2}CD$	4. ?
5. $AY = \frac{1}{2}AB$; $CZ = \frac{1}{2}CD$	5. ?
6. $AY = CZ$, or $\overline{AY} \cong \overline{CZ}$	6. ?
7. $\overline{OA} \cong \overline{OC}$	7. ?
8. rt. $\triangle OYA \cong$ rt. $\triangle OZC$	8. ?
9. $\overline{OY} \cong \overline{OZ}$, or $OY = OZ$	9. ?

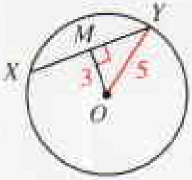
7. Suppose that in Theorem 9-6, the words "circle" and "circles" are replaced by "sphere" and "spheres." Is the resulting statement true?

Written Exercises

In the diagrams that follow, O is the center of the circle.

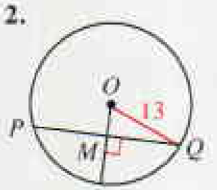
A

1.



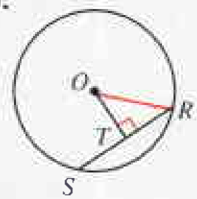
$XY = \underline{\ ?\ }$

2.



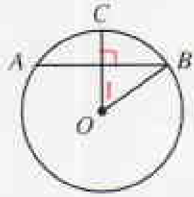
$PQ = 24$; $OM = \underline{\ ?\ }$

3.



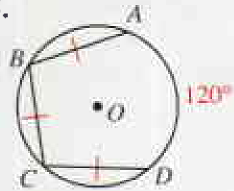
$OT = 9$; $RS = 18$
$OR = \underline{\ ?\ }$

4.



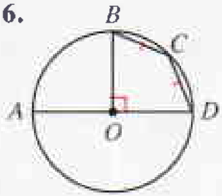
$m\widehat{ACB} = 110$;
$m\angle 1 = \underline{\ ?\ }$

5.



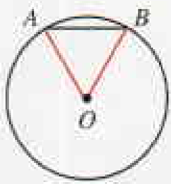
$m\widehat{BC} = \underline{\ ?\ }$

6.



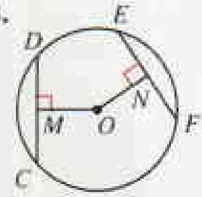
$m\widehat{CD} = \underline{\ ?\ }$

7.



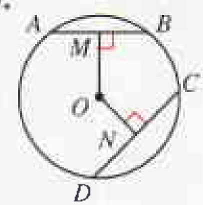
$m\angle AOB = 60$;
$AB = 24$; $OA = \underline{\ ?\ }$

8.



$OM = ON = 7$;
$CM = 6$; $EF = \underline{\ ?\ }$

9.



$AB = 18$; $OM = 12$;
$ON = 10$; $CD = \underline{\ ?\ }$

10. Sketch a circle with two noncongruent chords. Is the longer chord farther from the center or closer to the center than the shorter chord?

11. Sketch a circle O with radius 10 and chord $\overline{XY}$ 8 cm long. How far is the chord from O?

12. Sketch a circle Q with a chord $\overline{RS}$ that is 16 cm long and 2 cm from Q. What is the radius of $\odot Q$?

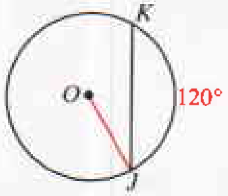
13. Sketch a circle P with radius 5 cm and chord $\overline{AB}$ that is 2 cm from P. Find the length of $\overline{AB}$.

14. Given: $\widehat{JZ} \cong \widehat{KZ}$
Prove: $\angle J \cong \angle K$

15. Prove the converse of Exercise 14.

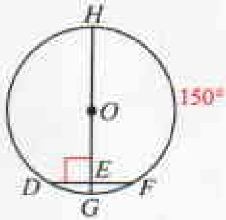
B **16.** Write a paragraph proof of part (2) of Theorem 9-4. First list what is given and what is to be proved.

17.



If $OJ = 10$, $JK = \underline{\ ?\ }$.

18.



If $OE = 8\sqrt{3}$, $HG = \underline{\ ?\ }$.

19. A plane 5 cm from the center of a sphere intersects the sphere in a circle with diameter 24 cm. Find the diameter of the sphere.

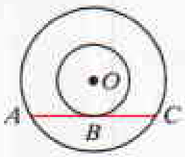
20. A plane P cuts sphere O in a circle that has diameter 20. If the diameter of the sphere is 30, how far is the plane from O?

21. Use trigonometry to find the measure of the arc cut off by a chord 12 cm long in a circle of radius 10 cm.

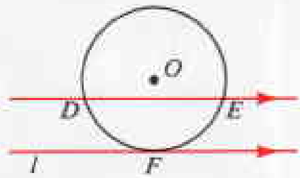
22. In $\odot O$, $m\overset{\frown}{RS} = 70$ and $RS = 20$. Use trigonometry to find the radius of $\odot O$.

State and prove a theorem suggested by the figure.

C **23.**



24.



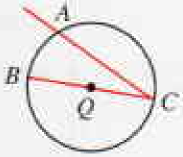
25. A, B, C are points on $\odot O$ such that $\triangle ABC$ is equilateral. If the radius of the circle is 6, what is the perimeter of $\triangle ABC$?

26. Investigate the possibility, given a circle, of drawing two chords whose lengths are in the ratio 1:2 and whose distances from the center are in the ratio 2:1. If the chords can be drawn, find the length of each in terms of the radius. If not, prove that the figure is impossible.

27. Three parallel chords of $\odot O$ are drawn as shown. Their lengths are 20, 16, and 12 cm. Find, to the nearest tenth of a centimeter, the length of chord $\overline{XY}$ (not shown).

Self-Test 1

1. Points A, B, and C lie on $\odot Q$.
  a. Name two radii of $\odot Q$.
  b. Name a diameter of $\odot Q$.
  c. Name a chord and a secant of $\odot Q$.



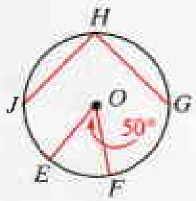
2. Sketch each of the following.
  a. $\triangle ABC$ inscribed in $\odot O$ **b.** Quad. $LUMX$ circumscribed about $\odot Q$

3. $\overline{NP}$ is tangent to $\odot O$ at P. If $NO = 25$ and $NP = 20$, find OP.

4. A plane passes through the common center of two concentric spheres. Describe the intersection of the plane and the two spheres.

5. Find the length of a chord that is 3 cm from the center of a circle with radius 7 cm.

6. Points E, F, G, H, and J lie on $\odot O$.
  a. $m\widehat{EF} =$ __?__ and $m\widehat{EHF} =$ __?__.
  b. Suppose $\overline{JH} \cong \overline{HG}$. State the theorem that supports the conclusion that $\widehat{JH} \cong \widehat{HG}$.



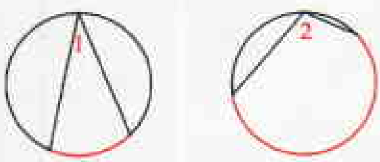
Angles and Segments

Objectives

1. Solve problems and prove statements involving inscribed angles.
2. Solve problems and prove statements involving angles formed by chords, secants, and tangents.
3. Solve problems involving lengths of chords, secant segments, and tangent segments.

9-5 *Inscribed Angles*

Angles 1 and 2 shown at the right are called *inscribed angles*. An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle. We say that the angles at the right *intercept* the arcs shown in color. $\angle 1$ intercepts a minor arc. $\angle 2$ intercepts a major arc.



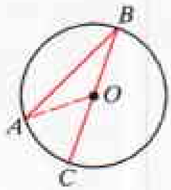
The next theorem compares the measure of an inscribed angle with the measure of its intercepted arc. Its proof requires us to consider three possible cases.

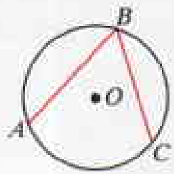
Theorem 9-7

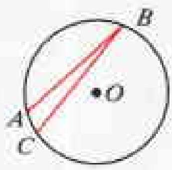
The measure of an inscribed angle is equal to half the measure of its intercepted arc.

Given: $\angle ABC$ inscribed in $\odot O$

Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$







Case I:
Point O lies on $\angle ABC$.

Case II:
Point O lies inside $\angle ABC$.

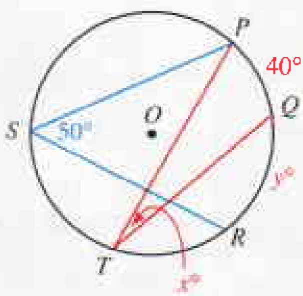
Case III:
Point O lies outside $\angle ABC$.

Key steps of proof of Case I:

1. Draw radius $\overline{OA}$ and let $m\angle ABC = x$.
2. $m\angle A = x$ (Why?)
3. $m\angle AOC = 2x$ (Why?)
4. $m\widehat{AC} = 2x$ (Why?)
5. $m\angle ABC = \frac{1}{2}m\widehat{AC}$ (Substitution Prop.)

Now that Case I has been proved, it can be used to prove Case II and Case III. An auxiliary line will be used in those proofs, which are left as Classroom Exercises 12 and 13.

Example 1 Find the values of x and y in $\odot O$.



Solution

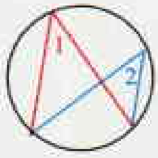
$m\angle PTQ = \frac{1}{2}m\widehat{PQ}$, so
$x = \frac{1}{2} \cdot 40 = 20.$

$m\angle PSR = \frac{1}{2}m\widehat{PR}$, so
$50 = \frac{1}{2}(40 + y)$ and $y = 60.$

Proofs of the following three corollaries of Theorem 9-7 will be considered in Classroom Exercises 1–3.

Corollary 1

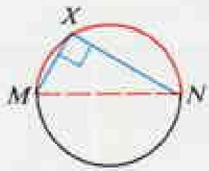
If two inscribed angles intercept the same arc, then the angles are congruent.



$\angle 1 \cong \angle 2$

Corollary 2

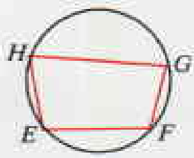
An angle inscribed in a semicircle is a right angle.



If $\overset{\frown}{MXN}$ is a semicircle, then $\angle X$ is a right angle.

Corollary 3

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.



$\angle E$ is supp. to $\angle G$.
$\angle F$ is supp. to $\angle H$.

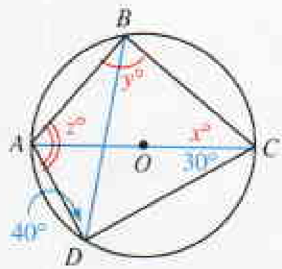
Example 2 Find the values of x, y, and z.

Solution $\angle ADB$ and $\angle ACB$ intercept the same arc, so $x = 40$. (Corollary 1)

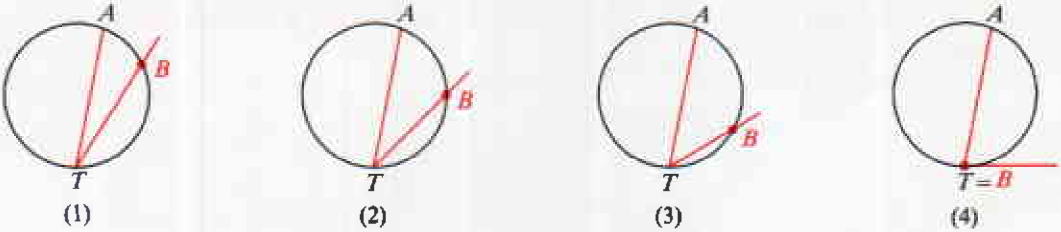
$\angle ABC$ is inscribed in a semicircle, so $\angle ABC$ is a right angle and $y = 90$. (Corollary 2)

$ABCD$ is an inscribed quadrilateral, so $\angle BAD$ and $\angle BCD$ are supplementary. (Corollary 3)

Therefore, $z = 180 - (x + 30)$
$z = 180 - (40 + 30) = 110.$



Study the diagrams below from left to right. Point B moves along the circle closer and closer to point T. Finally, in diagram (4), point B has merged with T, and one side of $\angle T$ has become a tangent.



Apply Theorem 9-7 to diagrams (1), (2), and (3) and you have $m\angle T = \frac{1}{2}m\widehat{AB}$. As you might expect, this equation applies to diagram (4), too, since we say that $\angle T$ intercepts $\widehat{AB}$ in this case as well. Diagram (4) suggests Theorem 9-8. In Exercises 13–15 you will prove the three cases of the theorem.

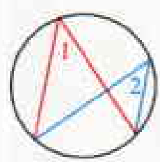
Theorem 9-8

The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.

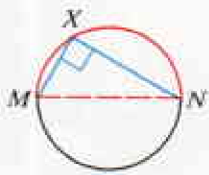
For example, if $\overline{PT}$ is tangent to the circle and $\overline{AT}$ is a chord, then $\angle ATP$ intercepts $\widehat{AT}$ and $m\angle ATP = \frac{1}{2}m\widehat{AT} = \frac{1}{2} \cdot 140 = 70$.

Classroom Exercises

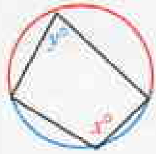
1. Explain why Corollary 1 of Theorem 9-7 is true. That is, explain why $\angle 1 \cong \angle 2$.



2. Explain why Corollary 2 is true. That is, explain how the fact that $\widehat{MXN}$ is a semicircle leads to a conclusion that $\angle X$ is a right angle.

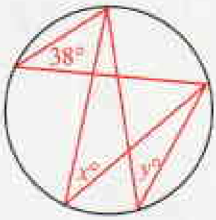


3. **a.** What is the sum of the measures of the red and blue arcs?
b. Explain how part (a) allows you to deduce that $x + y = 180$.
c. State the corollary of Theorem 9-7 that you have just proved.



Tangents and chords are shown. Find the values of x and y. In Exercise 5, O is the center of the circle.

4.



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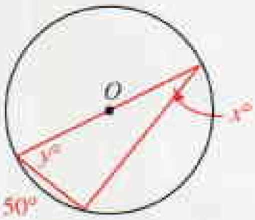

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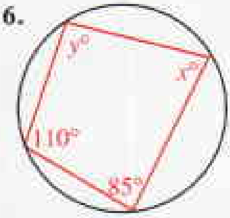

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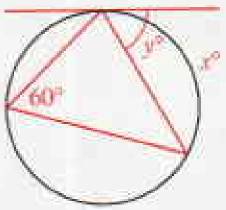
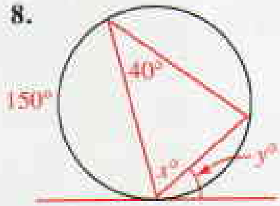


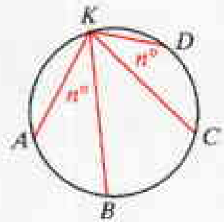
8.



9.

10. **a.** State the contrapositive of Corollary 3.
b. In quadrilateral $PQRS$, $m\angle P = 100$ and $m\angle R = 90$. Is it possible to circumscribe a circle about $PQRS$? Why or why not?

11. In the diagram, $m\angle AKB = m\angle CKD = n$. $m\widehat{AB} = \underline{\ ?\ }$ and $m\widehat{CD} = \underline{\ ?\ }$. State a theorem suggested by this exercise.



12. Outline a proof of Case II of Theorem 9-7. Use the diagram on page 350. (*Hint*: Draw the diameter from B and apply Case I.)

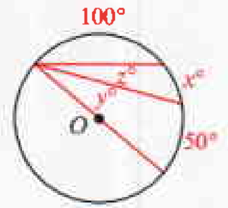
13. Repeat Exercise 12 for Case III.

14. Equilateral $\triangle ABC$ is inscribed in $\odot O$. Tangents to the circle at A and C meet at D. What kind of figure is $ABCD$?

Written Exercises

In the diagrams that follow, O is the center of the circle. In Exercises 1–9 find the values of x, y, and z.

A **1.**



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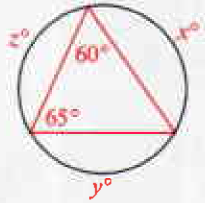

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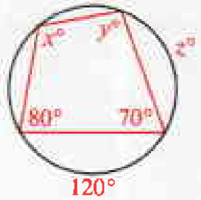

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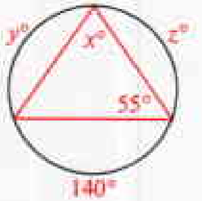

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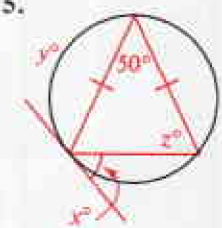

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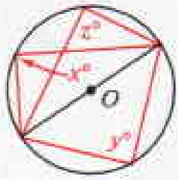

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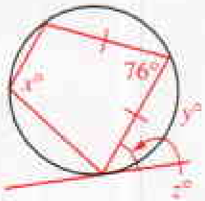
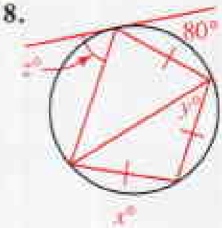
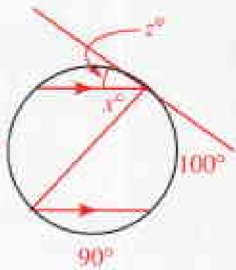


8.



9.

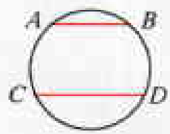


10. Prove: If two chords of a circle are parallel, the two arcs between the chords are congruent.

Given: $\overline{AB} \parallel \overline{CD}$

Prove: $\overset{\frown}{AC} \cong \overset{\frown}{BD}$

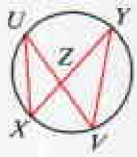
(*Hint*: Draw an auxiliary line.)



11. a. State the converse of the statement in Exercise 10.

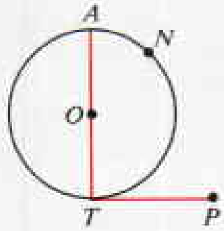
b. Is this converse true or false? If it is true, write a proof. If not, explain why it is false.

12. Prove: $\triangle UXZ \sim \triangle YVZ$



Exercises 13–15 prove the three possible cases of Theorem 9-8. In each case you are given chord $\overline{TA}$ and tangent $\overline{TP}$ of $\odot O$.

13. Supply reasons for the key steps of the proof that $m\angle ATP = \frac{1}{2}m\widehat{ANT}$ in Case I.

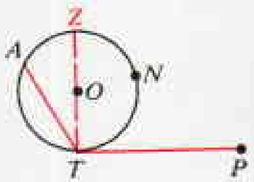


Case I: O lies on $\angle ATP$.

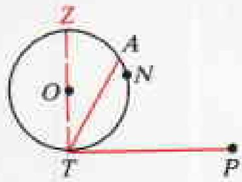
1. $\overline{TP} \perp \overline{TA}$ and $m\angle ATP = 90$.
2. $\widehat{ANT}$ is a semicircle and $m\widehat{ANT} = 180$.
3. $m\angle ATP = \frac{1}{2}m\widehat{ANT}$

In Case II and Case III, $\overline{AT}$ is not a diameter. You can draw diameter $\overline{TZ}$ and then use Case I, Theorem 9-7, and the Angle Addition and Arc Addition Postulates.

B **14.** Case II. O lies inside $\angle ATP$. Prove $m\angle ATP = \frac{1}{2}m\widehat{ANT}$



15. Case III. O lies outside $\angle ATP$. Prove $m\angle ATP = \frac{1}{2}m\widehat{ANT}$



16. Prove that if one pair of opposite sides of an inscribed quadrilateral are congruent, then the other sides are parallel.

17. Draw an inscribed quadrilateral $ABCD$ and its diagonals intersecting at E. Name two pairs of similar triangles.

18. Draw an inscribed quadrilateral $PQRS$ with shortest side $\overline{PS}$. Draw its diagonals intersecting at T. Extend $\overrightarrow{QP}$ and $\overrightarrow{RS}$ to meet at V. Name two pairs of similar triangles such that each triangle has a vertex at V.

Exercises 19–21 refer to a quadrilateral $ABCD$ inscribed in a circle.

19. $m\angle A = x$, $m\angle B = 2x$, and $m\angle C = x + 20$. Find x and $m\angle D$.

20. $m\angle A = x^2$, $m\angle B = 9x - 2$, and $m\angle C = 11x$. Find x and $m\angle D$.

21. $m\angle D = 75$, $m\widehat{AB} = x^2$, $m\widehat{BC} = 5x$, and $m\widehat{CD} = 6x$. Find x and $m\angle A$.

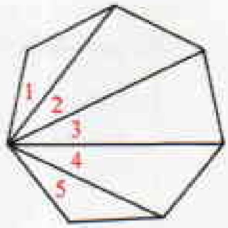
22. Parallelogram $ABCD$ is inscribed in $\odot O$. Find $m\angle A$.

23. Equilateral $\triangle ABC$ is inscribed in a circle. P and Q are midpoints of $\widehat{BC}$ and $\widehat{CA}$, respectively. What kind of figure is quadrilateral $AQPB$? Justify your answer.

24. The diagram at the right shows a regular polygon with 7 sides.

a. Explain why the numbered angles are all congruent. (*Hint*: You may assume that a circle can be circumscribed about any regular polygon.)

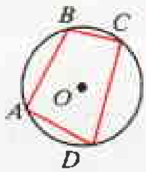
b. Will your reasoning apply to a regular polygon with any number of sides?



C **25.** Given: Vertices A, B, and C of quadrilateral $ABCD$ lie on $\odot O$; $m\angle A + m\angle C = 180$; $m\angle B + m\angle D = 180$.

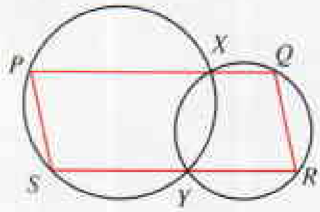
Prove: D lies on $\odot O$.

(*Hint*: Use an indirect proof. Assume temporarily that D is not on $\odot O$. You must then treat two cases: (1) D is inside $\odot O$, and (2) D is outside $\odot O$. In each case let X be the point where $\overrightarrow{AD}$ intersects $\odot O$ and draw $\overline{CX}$. Show that what you can conclude about $\angle AXC$ contradicts the given information.)



26. Given: $\overline{PQ} \parallel \overline{SR}$

Prove: $\overline{PS} \parallel \overline{QR}$



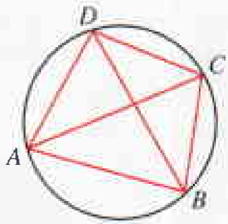
27. *Ptolemy's Theorem* states that in an inscribed quadrilateral, the sum of the products of its opposite sides is equal to the product of its diagonals. This means that for $ABCD$ shown,

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Prove the theorem by choosing point Q on $\overline{AC}$ so that $\angle ADQ \cong \angle BDC$. Then show $\triangle ADQ \sim \triangle BDC$ and $\triangle ADB \sim \triangle QDC$. Use these similar triangles to show that

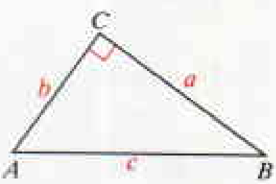
$$AQ = \frac{BC \cdot AD}{BD} \text{ and } QC = \frac{AB \cdot CD}{BD}.$$

Add these two equations and complete the proof.

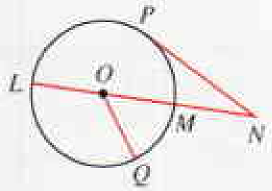


28. Equilateral $\triangle ABC$ is inscribed in a circle. P is any point on $\overset{\frown}{BC}$. Prove $PA = PB + PC$. (*Hint*: Use Ptolemy's Theorem.)

★ **29.** Angle C of $\triangle ABC$ is a right angle. The sides of the triangle have the lengths shown. The smallest circle (not shown) through C that is tangent to $\overline{AB}$ intersects $\overline{AC}$ at J and $\overline{BC}$ at K. Express the distance JK in terms of a, b, and c.



Mixed Review Exercises



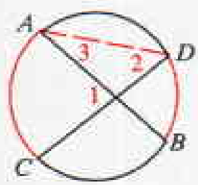
1. Name a diameter of $\odot O$.
2. Name a secant of $\odot O$.
3. Name a tangent segment.
4. If $OQ = 7$, then $LM = \underline{\ ?\ }$.
5. If $m\overset{\frown}{MQ} = x$, express $m\overset{\frown}{QLM}$ in terms of x.
6. Find the geometric mean between 4 and 9.

9-6 *Other Angles*

The preceding section dealt with angles that have their vertices on a circle. Theorem 9-9 deals with the angle formed by two chords that intersect inside a circle. Such an angle and its vertical angle intercept two arcs.

Theorem 9-9

The measure of an angle formed by two chords that intersect inside a circle is equal to half the sum of the measures of the intercepted arcs.



Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle.

Prove: $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$

Proof:

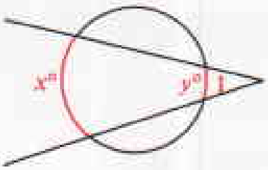
Statements	Reasons
1. Draw chord $\overline{AD}$.	1. Through any two points there is exactly one line.
2. $m\angle 1 = m\angle 2 + m\angle 3$	2. The measure of an exterior $\angle$ of a $\triangle$ = the sum of the measures of the two remote interior $\angle$s.
3. $m\angle 2 = \frac{1}{2}m\overset{\frown}{AC}$; $m\angle 3 = \frac{1}{2}m\overset{\frown}{BD}$	3. The measure of an inscribed angle is equal to half the measure of its intercepted arc.
4. $m\angle 1 = \frac{1}{2}m\overset{\frown}{AC} + \frac{1}{2}m\overset{\frown}{BD}$ or $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$	4. Substitution (Step 3 in Step 2)

One case of the next theorem will be proved in Classroom Exercise 10, the other two cases in Exercises 25 and 26. Notice that angles formed by two secants, two tangents, or a secant and a tangent intercept two arcs.

Theorem 9-10

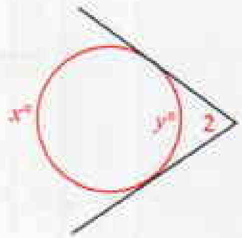
The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside a circle is equal to half the difference of the measures of the intercepted arcs.

Case I: Two secants



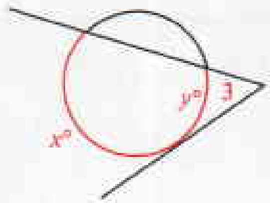
$m\angle 1 = \frac{1}{2}(x - y)$

Case II: Two tangents



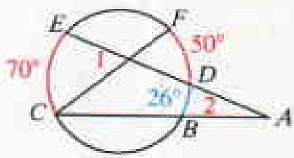
$m\angle 2 = \frac{1}{2}(x - y)$

Case III: A secant and a tangent



$m\angle 3 = \frac{1}{2}(x - y)$

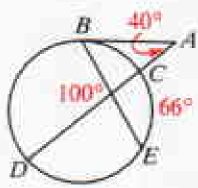
Example 1 Find the measures of $\angle 1$ and $\angle 2$.



Solution

$m\angle 1 = \frac{1}{2}(m\overset{\frown}{CE} + m\overset{\frown}{FD})$ (Theorem 9-9)
$m\angle 1 = \frac{1}{2}(70 + 50) = 60$
$m\angle 2 = \frac{1}{2}(m\overset{\frown}{CE} - m\overset{\frown}{BD})$ (Theorem 9-10)
$m\angle 2 = \frac{1}{2}(70 - 26) = 22$

Example 2 $\overline{BA}$ is a tangent. Find $m\overset{\frown}{BD}$ and $m\overset{\frown}{BC}$.



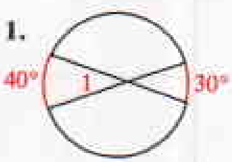
Solution

$100 = \frac{1}{2}(m\overset{\frown}{BD} + 66)$ (Theorem 9-9)
$200 = m\overset{\frown}{BD} + 66$, so $m\overset{\frown}{BD} = 134$
$40 = \frac{1}{2}(m\overset{\frown}{BD} - m\overset{\frown}{BC})$ (Theorem 9-10)
$80 = 134 - m\overset{\frown}{BC}$, so $m\overset{\frown}{BC} = 54$

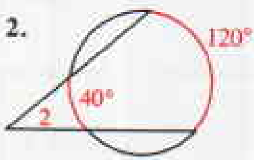
Classroom Exercises

Find the measure of each numbered angle.

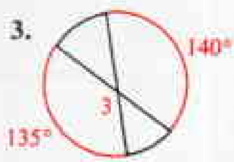
1.



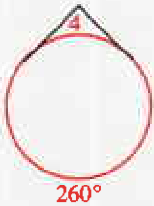
2.



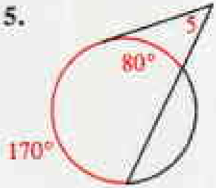
3.



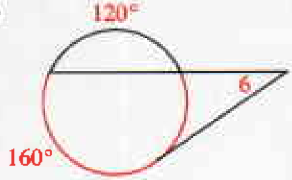
4.



5.

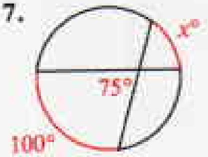


6.

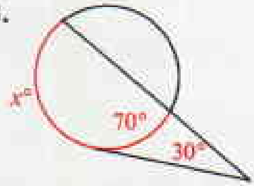


State an equation you could use to find the value of x. Then solve for x.

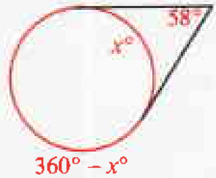
7.



8.



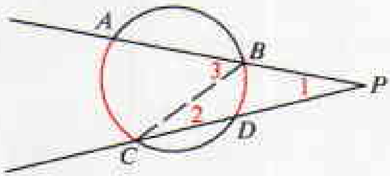
9.



10. Supply reasons to complete a proof of Case I of Theorem 9-10.

Given: Secants $\overline{PA}$ and $\overline{PC}$

Prove: $m\angle 1 = \frac{1}{2}(m\widehat{AC} - m\widehat{BD})$

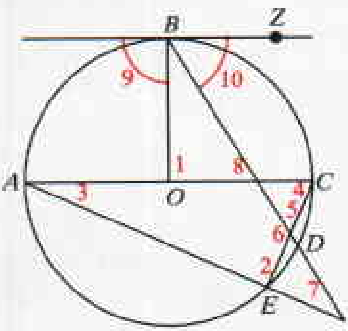


Proof:

1. Draw chord $\overline{BC}$.
2. $m\angle 1 + m\angle 2 = m\angle 3$
3. $m\angle 1 = m\angle 3 - m\angle 2$
4. $m\angle 3 = \frac{1}{2}m\widehat{AC}$; $m\angle 2 = \frac{1}{2}m\widehat{BD}$
5. $m\angle 1 = \frac{1}{2}m\widehat{AC} - \frac{1}{2}m\widehat{BD}$, or $m\angle 1 = \frac{1}{2}(m\widehat{AC} - m\widehat{BD})$

Written Exercises

A **1–10.** $\overleftrightarrow{BZ}$ is tangent to $\odot O$; $\overline{AC}$ is a diameter; $m\widehat{BC} = 90$; $m\widehat{CD} = 30$; $m\widehat{DE} = 20$. Draw your own large diagram so that you can write arc measures alongside the arcs. Find the measure of each numbered angle.



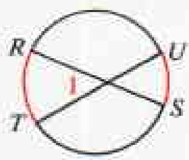
Complete.

11. If $m\widehat{RT} = 80$ and $m\widehat{US} = 40$, then $m\angle 1 = \underline{\ ?\ }$.

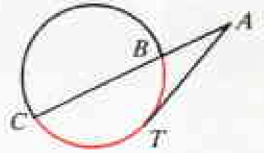
12. If $m\widehat{RU} = 130$ and $m\widehat{TS} = 100$, then $m\angle 1 = \underline{\ ?\ }$.

13. If $m\angle 1 = 50$ and $m\widehat{RT} = 70$, then $m\widehat{US} = \underline{\ ?\ }$.

14. If $m\angle 1 = 52$ and $m\widehat{US} = 36$, then $m\widehat{RT} = \underline{\ ?\ }$.

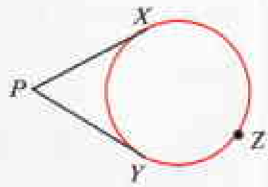


In Exercises 15–17 $\overline{AT}$ is a tangent.



15. If $m\overset{\frown}{CT} = 110$ and $m\overset{\frown}{BT} = 50$, then $m\angle A =$ _?_.
16. If $m\angle A = 28$ and $m\overset{\frown}{BT} = 46$, then $m\overset{\frown}{CT} =$ _?_.
17. If $m\angle A = 35$ and $m\overset{\frown}{CT} = 110$, then $m\overset{\frown}{BT} =$ _?_.

In Exercises 18–21 $\overline{PX}$ and $\overline{PY}$ are tangents.



18. If $m\overset{\frown}{XZY} = 250$, then $m\angle P =$ _?_.
19. If $m\overset{\frown}{XY} = 90$, then $m\angle P =$ _?_.
20. If $m\overset{\frown}{XY} = t$, then $m\overset{\frown}{XZY} =$ _?_ and $m\angle P =$ _?_ in terms of t.
21. If $m\angle P = 65$, then $m\overset{\frown}{XY} =$ _?_.

B 22. A secant and a tangent to a circle intersect in a 42° angle. The two arcs of the circle intercepted by the secant and tangent have measures in a 7:3 ratio. Find the measure of the third arc.

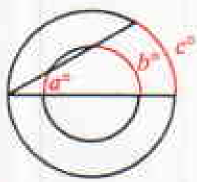
23. A quadrilateral circumscribed about a circle has angles of 80°, 90°, 94°, and 96°. Find the measures of the four nonoverlapping arcs determined by the points of tangency.

24. In the inscribed quadrilateral $ABCD$, the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are congruent. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ meet at a 32° angle. Find the measures of the angles of $ABCD$.

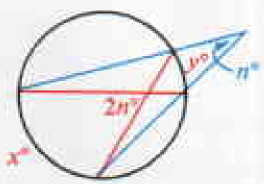
25. Prove Case II of Theorem 9-10. (*Hint*: See Classroom Exercise 10. Draw a figure like the second one shown below the theorem on page 358. Label your figure, and draw the chord joining the points of tangency.)

26. Prove Case III of Theorem 9-10.

27. Write an equation involving a, b, and c.

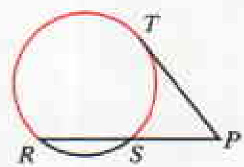


28. Find the ratio $x:y$.

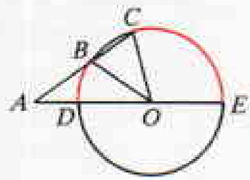


29. Isosceles $\triangle ABC$ with base $\overline{BC}$ is inscribed in a circle. P is a point on $\overset{\frown}{AC}$ and $\overrightarrow{AP}$ and $\overrightarrow{BC}$ meet at Q. Prove that $\angle ABP \cong \angle Q$.

C 30. $\overline{PT}$ is a tangent. It is known that $80 < m\overset{\frown}{RS} < m\overset{\frown}{ST} < 90$. State as much as you can about the measure of $\angle P$.



31. $\overline{AC}$ and $\overline{AE}$ are secants of $\odot O$. It is given that $\overline{AB} \cong \overline{OB}$. Discover and prove a relation between the measures of $\overset{\frown}{CE}$ and $\overset{\frown}{BD}$.



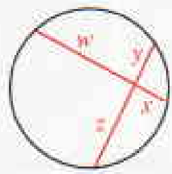
32. Take any point P outside a circle. Draw a tangent segment $\overline{PT}$ and a secant $\overline{PBA}$ with A and B points on the circle. Take K on $\overline{PA}$ so that $PK = PT$. Draw $\overrightarrow{TK}$. Let the intersection of $\overrightarrow{TK}$ with the circle be point X. Discover and prove a relationship between $\overset{\frown}{AX}$ and $\overset{\frown}{XB}$.

Explorations

These exploratory exercises can be done using a computer with a program that draws and measures geometric figures.

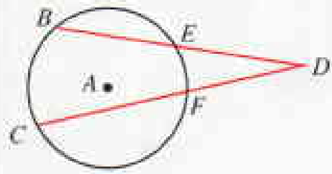
1. Draw a circle. Choose four points on the circle. Draw two intersecting chords using the points as endpoints.

   Measure the lengths of the pieces of the chords and compute the products $w \cdot x$ and $y \cdot z$. What do you notice?



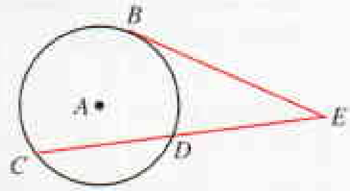
2. Draw any circle A. Choose two points B and C on the circle and a point D outside the circle. Draw secants $\overline{BD}$ and $\overline{CD}$. Label their intersections with the circle as E and F.

   Measure and compute $DE \cdot DB$ and $DF \cdot DC$. What do you notice?



3. Draw any circle A. Choose three points B, C, and D on the circle. Draw a tangent to the circle through point B that intersects $\overleftrightarrow{CD}$ at a point E.

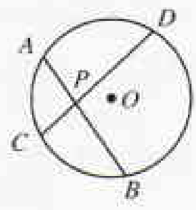
   Measure and compute $(BE)^2$ and $ED \cdot CE$. What do you notice?



9-7 *Circles and Lengths of Segments*

You can use similar triangles to prove that lengths of chords, secants, and tangents are related in interesting ways.

In the figure at the right, chords $\overline{AB}$ and $\overline{CD}$ intersect inside $\odot O$. $\overline{AP}$ and $\overline{PB}$ are called *the segments of chord* $\overline{AB}$. As we did with the terms "radius" and "diameter" we will use the phrase "segment of a chord" to refer to the length of a segment as well as to the segment itself.

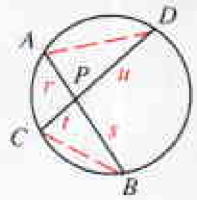


Theorem 9-11

When two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other chord.

Given: $\overline{AB}$ and $\overline{CD}$ intersect at P.

Prove: $r \cdot s = t \cdot u$



Proof:

Statements	Reasons
1. Draw chords $\overline{AD}$ and $\overline{CB}$.	1. Through any two points there is exactly one line.
2. $\angle A \cong \angle C$; $\angle D \cong \angle B$	2. If two inscribed angles intercept __?__.
3. $\triangle APD \sim \triangle CPB$	3. __?__
4. $\frac{r}{t} = \frac{u}{s}$	4. __?__
5. $r \cdot s = t \cdot u$	5. A property of proportions

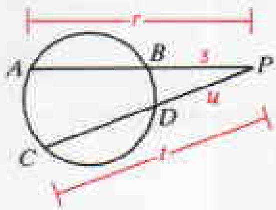
For a proof of the following theorem, see Classroom Exercise 7. In the diagram for the theorem, $\overline{AP}$ and $\overline{CP}$ are *secant segments*. $\overline{BP}$ and $\overline{DP}$ are exterior to the circle and are referred to as *external segments*. The terms ''secant segment'' and ''external segment'' can refer to the length of a segment as well as to the segment itself.

Theorem 9-12

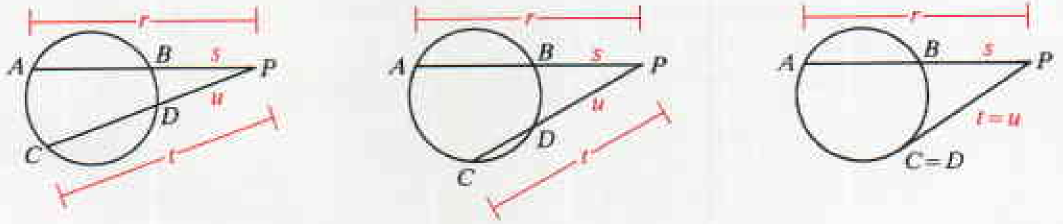
When two secant segments are drawn to a circle from an external point, the product of one secant segment and its external segment equals the product of the other secant segment and its external segment.

Given: $\overline{PA}$ and $\overline{PC}$ drawn to the circle from point P

Prove: $r \cdot s = t \cdot u$



Study the diagrams at the top of the next page from left to right. As $\overline{PC}$ approaches a position of tangency, C and D move closer together until they merge. Then $\overline{PC}$ becomes a tangent, and $t = u$.

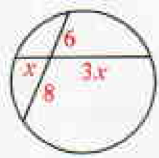


In the first two diagrams we know that $r \cdot s = t \cdot u$. In the third diagram, u and t both become equal to the length of the tangent segment, and the equation becomes $r \cdot s = t^2$. This result, stated below, will be proved more formally in Exercise 10. As with earlier terms, the term "tangent segment" can refer to the length of a segment as well as to the segment itself.

Theorem 9-13

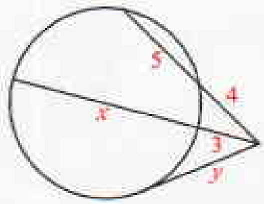
When a secant segment and a tangent segment are drawn to a circle from an external point, the product of the secant segment and its external segment is equal to the square of the tangent segment.

Example 1 Find the value of x.



Solution $3x \cdot x = 6 \cdot 8$ (Theorem 9-11)
$3x^2 = 48$, $x^2 = 16$, and $x = 4$

Example 2 Find the values of x and y.

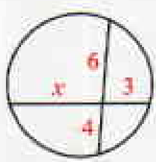


Solution $4(4 + 5) = 3(3 + x)$ (Theorem 9-12)
$36 = 3(3 + x)$, $12 = 3 + x$, and $x = 9$
$4(4 + 5) = y^2$ (Theorem 9-13)
$36 = y^2$, so $y = 6$

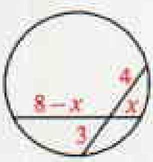
Classroom Exercises

Chords, secants, and tangents are shown. State the equation you would use to find the value of x. Then solve for x.

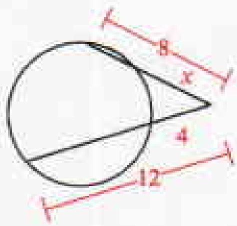
1.



2.

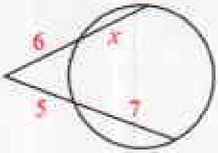


3.



Chords, secants, and tangents are shown. State the equation you would use to find the value of x. Then solve for x.

4.



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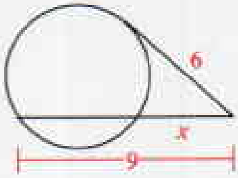

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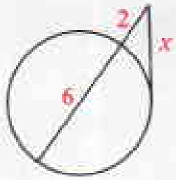
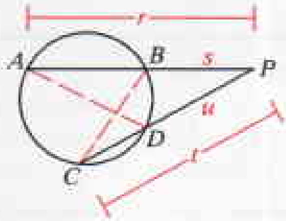


7. Supply reasons to complete the proof of Theorem 9-12.

Given: $\overline{PA}$ and $\overline{PC}$ drawn to the circle from point P

Prove: $r \cdot s = t \cdot u$

Proof:

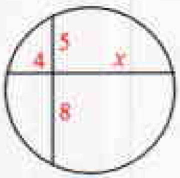


1. Draw chords $\overline{AD}$ and $\overline{BC}$.
2. $\angle A \cong \angle C$
3. $\angle P \cong \angle P$
4. $\triangle APD \sim \triangle CPB$
5. $\frac{r}{t} = \frac{u}{s}$
6. $r \cdot s = t \cdot u$

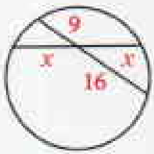
Written Exercises

Chords, secants, and tangents are shown. Find the value of x.

A **1.**



2.



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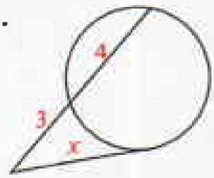

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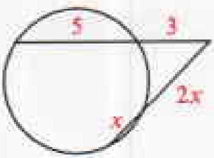

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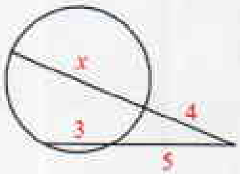

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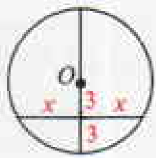

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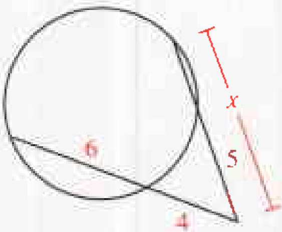

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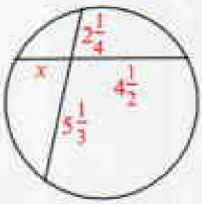
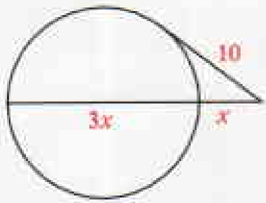


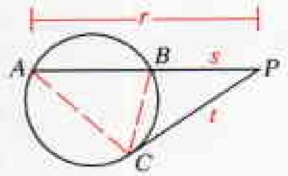
9.



10. Write a proof of Theorem 9-13.

Given: Secant segment $\overline{PA}$ and tangent segment $\overline{PC}$ drawn to the circle from P.

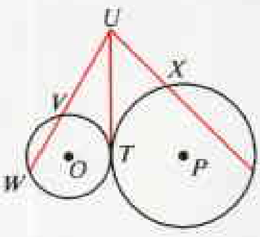
Prove: $r \cdot s = t^2$



Plan for Proof: Draw chords $\overline{AC}$ and $\overline{BC}$. Show that $\angle A$ and $\angle PCB$ are congruent because they intercept the same arc. Then show that $\triangle PAC$ and $\triangle PCB$ are similar triangles and use the properties of proportions to complete the proof.

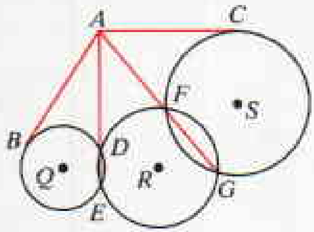
B **11.** Given: $\odot O$ and $\odot P$ are tangent to $\overleftrightarrow{UT}$ at T.

Prove: $UV \cdot UW = UX \cdot UY$



12. Given: $\overline{AB}$ is tangent to $\odot Q$; $\overline{AC}$ is tangent to $\odot S$.

Prove: $\overline{AB} \cong \overline{AC}$

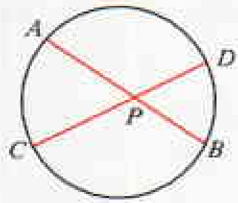


Chords $\overline{AB}$ and $\overline{CD}$ intersect at P. Find the lengths indicated.

Example $AP = 5$; $BP = 4$; $CD = 12$; $CP =$ __?__

Solution Let $CP = x$. Then $DP = 12 - x$.

$x(12 - x) = 5 \cdot 4$
$12x - x^2 = 20$
$x^2 - 12x + 20 = 0$
$(x - 2)(x - 10) = 0$
$x = 2$ or $x = 10$
$CP = 2$ or 10



13. $AP = 6$; $BP = 8$; $CD = 16$; $DP =$ __?__

14. $CD = 10$; $CP = 6$; $AB = 11$; $AP =$ __?__

15. $AB = 12$; $CP = 9$; $DP = 4$; $BP =$ __?__

16. $AP = 6$; $BP = 5$; $CP = 3 \cdot DP$; $DP =$ __?__

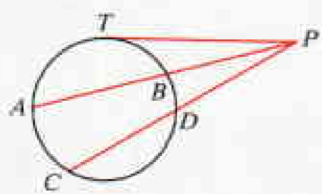
$\overline{PT}$ is tangent to the circle. Find the lengths indicated.

17. $PT = 6$; $PB = 3$; $AB =$ __?__

18. $PT = 12$; $CD = 18$; $PC =$ __?__

19. $PD = 5$; $CD = 7$; $AB = 11$; $PB =$ __?__

20. $PB = AB = 5$; $PD = 4$; $PT =$ __?__ and $PC =$ __?__

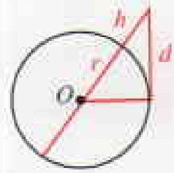


21. A secant, a radius, and a tangent of $\odot O$ are shown.

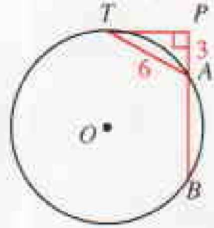
a. Explain why $(r + h)^2 = r^2 + d^2$.

b. Simplify the equation in part (a) to show that $d^2 = h(2r + h)$.

c. You have proved a special case of a theorem. What theorem is this?



22. $\overline{PT}$ is tangent to $\odot O$. Secant $\overrightarrow{BA}$ is perpendicular to $\overline{PT}$ at P. If $TA = 6$ and $PA = 3$, find **(a)** AB, **(b)** the distance from O to $\overline{AB}$, and **(c)** the radius of $\odot O$.

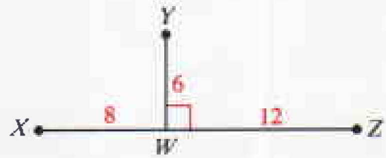


23. A bridge over a river has the shape of a circular arc. The span of the bridge is 24 meters. (The span is the length of the chord of the arc.) The midpoint of the arc is 4 meters higher than the endpoints. What is the radius of the circle that contains this arc?

24. A circle can be drawn through points X, Y, and Z.

a. What is the radius of the circle?

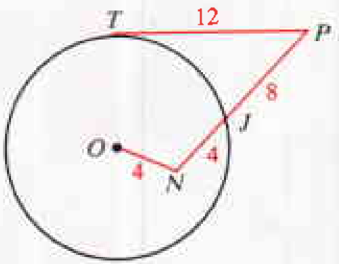
b. How far is the center of the circle from point W?



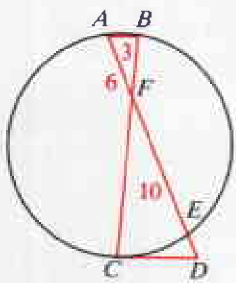
25. Draw two intersecting circles with common chord $\overline{PQ}$ and let X be any point on $\overline{PQ}$. Through X draw any chord $\overline{AB}$ of one circle. Also draw through X any chord $\overline{CD}$ of the other circle. Prove that $AX \cdot XB = CX \cdot XD$.

26. A line is tangent to two intersecting circles at P and Q. The common chord is extended to meet $\overline{PQ}$ at T. Prove that T is the midpoint of $\overline{PQ}$.

C 27. In the diagram at the left below, $\overline{PT}$ is tangent to $\odot O$ and $\overline{PN}$ intersects $\odot O$ at J. Find the radius of the circle.



Ex. 27

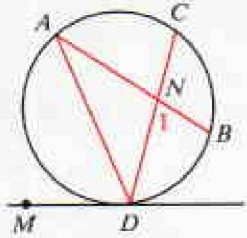


Ex. 28

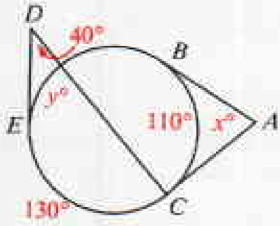
★ **28.** In the diagram at the right above, $\overline{CD}$ is a tangent, $\widehat{AC} \cong \widehat{BC}$, $AB = 3$, $AF = 6$, and $FE = 10$. Find ED. (*Hint*: Let $ED = x$ and $CD = y$. Then write two equations in x and y.)

Self-Test 2

$\overleftrightarrow{MD}$ is tangent to the circle.

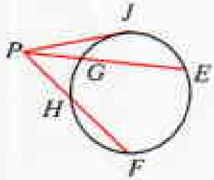


1. If $m\overset{\frown}{BD} = 80$, then $m\angle A =$ _?_.
2. If $m\angle ADM = 75$, then $m\overset{\frown}{AD} =$ _?_.
3. If $m\overset{\frown}{BD} = 80$ and $m\angle 1 = 81$, then $m\overset{\frown}{AC} =$ _?_.
4. If $AN = 12$, $BN = 6$, and $CN = 8$, then $DN =$ _?_.



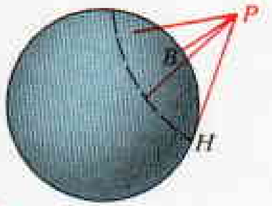
5. $\overline{AB}$, $\overline{AC}$, and $\overline{DE}$ are tangents.
   Find the values of x and y.

$\overline{PE}$ and $\overline{PF}$ are secants and $\overline{PJ}$ is a tangent.

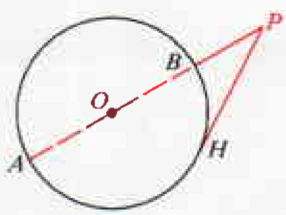


6. If $m\overset{\frown}{EF} = 100$ and $m\overset{\frown}{GH} = 30$, then $m\angle FPE =$ _?_.
7. If $PG = 4$, $PE = 15$, and $PH = 6$, then $PF =$ _?_.
8. If $PH = 8$ and $HF = 10$, then $PJ =$ _?_.

Application — *Distance to the Horizon*



If you look out over the surface of the Earth from a position at P, directly above point B on the surface, you see the horizon wherever your line of sight is tangent to the surface of the Earth. If the surface around B is smooth (say you are on the ocean on a calm day), the horizon will be a circle, and the higher your lookout is, the farther away this horizon circle will be.



You can use Theorem 9-13 to derive a formula that tells how far you can see from any given height. The diagram at the right shows a section through the Earth containing P, H, and O, the center of the Earth. $\overline{PH}$ is tangent to circle O at H. $\overline{PA}$ is a secant passing through the center O. Theorem 9-13 says that:

$$(PH)^2 = PA \cdot PB$$

In the formula $(PH)^2 = AP \cdot BP$, PH is the distance from the observer to the horizon, and BP is the observer's height above the surface of the Earth. If the height is small compared to the diameter, AB, of the Earth, then $AP \approx AB$ in the formula. Using 12,800,000 m for AB, you can rewrite the formula as:

$$(\text{distance})^2 \approx (12{,}800{,}000)(\text{height})$$

Taking square roots, you get:

$$\text{distance} \approx \sqrt{12{,}800{,}000} \cdot \sqrt{\text{height}} \approx 3600\sqrt{\text{height}}$$

So the approximate distance (in meters) to the horizon is 3600 times the square root of your height (in meters) above the surface of the Earth. If your height is less than 400 km, the error in this approximation will be less than one percent.

Exercises

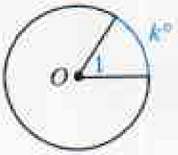
In Exercises 1 and 2 give your answer to the nearest kilometer, in Exercises 3 and 5 to the nearest 10 km, and in Exercise 4 to the nearest meter.

1. If you stand on a dune with your eyes about 16 m above sea level, how far out to sea can you look?
2. A lookout climbs high in the rigging of a sailing ship to a point 36 m above the water line. About how far away is the horizon?
3. From a balloon floating 10 km above the ocean, how far away is the farthest point you can see on the Earth's surface?
4. How high must a lookout be to see an object on the horizon 8 km away?
5. You are approaching the coast of Japan in a small sailboat. The highest point on the central island of Honshu is the cone of Mount Fuji, 3776 m above sea level. Roughly how far away from the mountain will you be when you can first see the top? (Assume that the sky is clear!)

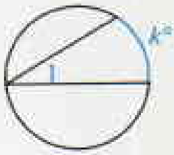
Chapter Summary

1. Many of the terms used with circles and spheres are discussed on pages 329 and 330.
2. If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency. The converse is also true.
3. Tangents to a circle from a point are congruent.

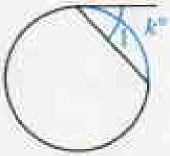
4. In the same circle or in congruent circles:
   a. Congruent minor arcs have congruent central angles.
      Congruent central angles have congruent arcs.
   b. Congruent arcs have congruent chords.
      Congruent chords have congruent arcs.
   c. Chords equally distant from the center are congruent.
      Congruent chords are equally distant from the center.
5. A diameter that is perpendicular to a chord bisects the chord and its arc.
6. If two inscribed angles intercept the same arc, then the angles are congruent.
7. An angle inscribed in a semicircle is a right angle.
8. If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.
9. Relationships expressed by formulas:



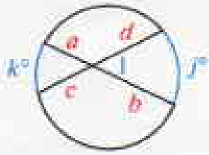
$m\angle 1 = k$



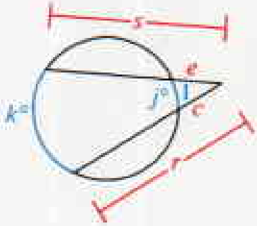
$m\angle 1 = \frac{1}{2}k$



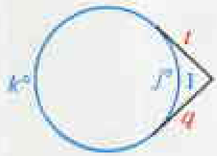
$m\angle 1 = \frac{1}{2}k$



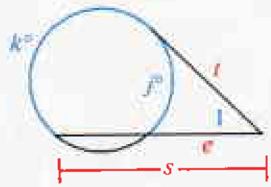
$m\angle 1 = \frac{1}{2}(k + j)$
$a \cdot b = c \cdot d$



$m\angle 1 = \frac{1}{2}(k - j)$
$s \cdot e = r \cdot c$



$m\angle 1 = \frac{1}{2}(k - j)$
$t = q$



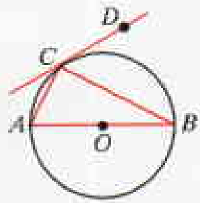
$m\angle 1 = \frac{1}{2}(k - j)$
$s \cdot e = t^2$

Chapter Review

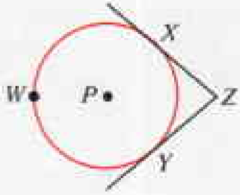
Points A, B, and C lie on $\odot O$.

9–1

1. $\overline{AC}$ is called a _?_, while $\overleftrightarrow{AC}$ is called a _?_.
2. $\overline{OB}$ is called a _?_.
3. The best name for $\overline{AB}$ is _?_.
4. $\triangle ABC$ is ______?______ (inscribed in/circumscribed about) $\odot O$.
5. $\overleftrightarrow{CD}$ intersects $\odot O$ in one point. $\overleftrightarrow{CD}$ is called a _?_.

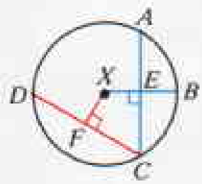


Lines $\overleftrightarrow{ZX}$ and $\overleftrightarrow{ZY}$ are tangent to $\odot P$.



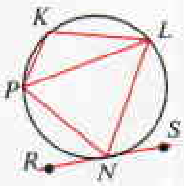
6. $\overline{PX}$, if drawn, would be __?__ to $\overleftrightarrow{XZ}$. 9–2
7. If the radius of $\odot P$ is 6 and $XZ = 8$, then $PZ =$ __?__.
8. If $m\angle Z = 90$ and $XZ = 13$, then $XY =$ __?__.
9. If $m\angle XPY = 100$, then $m\widehat{XY} =$ __?__. 9–3
10. If $m\widehat{XW} = 135$ and $m\widehat{WY} = 125$, then $m\widehat{XWY} =$ __?__.
11. If $\widehat{XW} \cong \widehat{WY}$, then $\angle XPW \cong$ __?__.

In $\odot X$, $m\widehat{AC} = 120$. 9–4



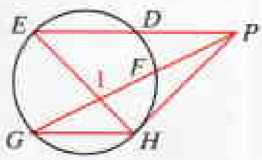
12. $m\widehat{AB} =$ __?__
13. If $\overline{AC} \cong \overline{CD}$, then $m\widehat{CD} =$ __?__.
14. If $XE = 5$ and $AC = 24$, then the radius = __?__.
15. If $\overline{AC} \cong \overline{DC}$, state the theorem that allows you to deduce that $XE = XF$.

$\overleftrightarrow{RS}$ is tangent to the circle at N. 9–5



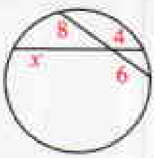
16. If $m\angle K = 105$, then $m\angle PNL =$ __?__.
17. If $m\widehat{PN} = 100$, then $m\angle PLN =$ __?__ and $m\angle PNR =$ __?__.
18. If $m\angle K = 110$, then $m\widehat{PNL} =$ __?__ and $m\widehat{PL} =$ __?__.

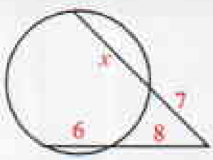
9–6

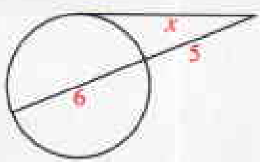


19. If $m\widehat{EF} = 120$ and $m\widehat{GH} = 90$, then $m\angle 1 =$ __?__.
20. If $m\widehat{EG} = 100$ and $m\widehat{DF} = 40$, then $m\angle EPG =$ __?__.
21. If $\overline{PH}$ is a tangent, $m\widehat{GH} = 90$ and $m\angle GPH = 25$, then $m\widehat{FH} =$ __?__.

Chords, secants, and a tangent are shown. Find the value of x. 9–7

22. 

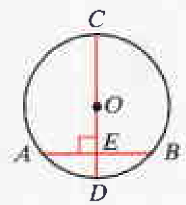
23. 

24. 

Chapter Test

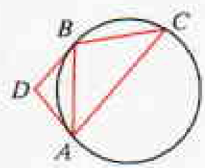
Classify each statement as true or false.

1. Opposite angles of an inscribed quadrilateral must be congruent.
2. If a chord in one circle is congruent to a chord in another circle, the arcs of these chords must have congruent central angles.
3. A diameter that is perpendicular to a chord must bisect the chord.
4. If a line bisects a chord, that line must pass through the center of the circle.
5. If $\overrightarrow{GM}$ intersects a circle in just one point, $\overleftrightarrow{GM}$ must be tangent to the circle.
6. It is possible to draw two circles so that no common tangents can be drawn.
7. An angle inscribed in a semicircle must be a right angle.
8. When one chord is farther from the center of a circle than another chord, the chord farther from the center is the longer of the two chords.

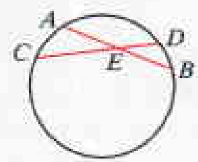


9. In $\odot O$, if $m\widehat{AB} = 100$, then $m\widehat{AC} =$ _?_.
10. If the radius of $\odot O$ is 17 and $AB = 30$, then $OE =$ _?_.

$\overline{DA}$ and $\overline{DB}$ are tangent to the circle.

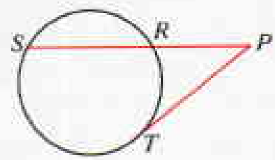


11. If $\overline{AB} \cong \overline{BC}$ and $m\widehat{BC} = 80$, then $m\angle ABC =$ _?_.
12. If $m\angle D = 110$, then $m\angle BCA =$ _?_.
13. Given: $m\widehat{BC} = m\widehat{AB}$
    Prove: $\overline{AC} \parallel \overline{DB}$



14. If $m\widehat{AC} = 40$ and $m\widehat{BD} = 28$, then $m\angle AEC =$ _?_.
15. If $AE = 10$, $EB = 9$, and $CE = 15$, then $ED =$ _?_.

$\overline{PT}$ is tangent to the circle.

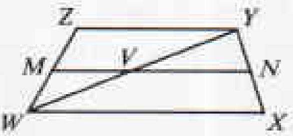


16. If $m\widehat{RS} = 120$ and $m\widehat{ST} = 160$, then $m\angle P =$ _?_.
17. If $PT = 12$ and $PS = 18$, then $PR =$ _?_.
18. Given: $\square ABCD$ is inscribed in a circle.
    Prove: $ABCD$ is a rectangle.

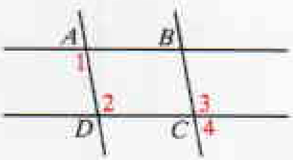
Cumulative Review: Chapters 1–9

A

1. If x, $x + 3$, and y are the lengths of the sides of a triangle, then $\underline{\ ?\ } < y < \underline{\ ?\ }$.
2. Find the measure of an angle if the measures of a supplement and a complement of the angle have the ratio 5:2.
3. Given: $\overline{MN}$ is the median of a trapezoid $WXYZ$. Prove: $\overline{MN}$ bisects $\overline{WY}$.



4. Prove: The diagonals of a rhombus divide the rhombus into four congruent triangles.
5. A 30°-60°-90° triangle is inscribed in a circle of radius 7. Find the length of each leg of the triangle.
6. Must three parallel lines be coplanar? Draw a diagram to illustrate your answer.
7. The measures of the angles of a triangle are in the ratio 1:9:10. Find the measure of each angle.
8. If a regular polygon has 18 sides, find the measure of each interior angle and the measure of each exterior angle.
9. If $ABCE$ is a square and $AC = 4$, find AB.
10. If the lengths of two sides of a right triangle are 6 and 10, find two possible lengths for the third side.
11. Given: $\angle 1 \cong \angle 2$; $\angle 2 \cong \angle 3$ Prove: $\overline{AB} \cong \overline{DC}$



12. When the altitude to the hypotenuse of a certain right triangle is drawn, the altitude divides the hypotenuse into segments of lengths 8 and 10. Find the length of the shorter leg.
13. Write **(a)** the contrapositive and **(b)** the inverse of the following statement: If quad. $ABCD$ is a parallelogram, then $\angle A \cong \angle C$.
14. If $\overrightarrow{OB}$ bisects $\angle AOC$, $m\angle AOB = 5t - 7$, and $m\angle AOC = 8t + 10$, find the numerical measure of $\angle BOC$.
15. Two chords of a circle intersect inside a circle, dividing one chord into segments of length 15 and 12 and the other chord into segments of length 9 and t. Find the value of t.
16. If points R and S on a number line have coordinates -11 and 3, and $\overline{RS}$ has midpoint T, find RS and ST.

17. Complete with *outside*, *inside*, or *on*: In a right triangle, (**a**) the medians intersect __?__ the triangle, (**b**) the altitudes intersect __?__ the triangle, and (**c**) the perpendicular bisectors of the sides intersect __?__ the triangle.

18. In $\triangle RST$, the bisector of $\angle T$ meets $\overline{RS}$ at X. $RS = 15$, $ST = 27$, $TR = 18$. Find RX.

19. Given: All of Bill's sisters like to dance.
What can you conclude from each additional statement? If no conclusion is possible, write *no conclusion*.

a. Janice is Bill's sister.
b. Holly loves to dance.
c. Maureen is not Bill's sister.
d. Kim does not like to dance.

20. Suppose someone plans to write an indirect proof of the statement "In $\square ABCD$ if $\overline{AB} \perp \overline{BC}$, then $ABCD$ is a rectangle." Write a correct first sentence of the indirect proof.

Complete each statement with the words *always*, *sometimes*, or *never*.

21. A contrapositive of a true conditional statement is __?__ true.

22. The sides of a triangle are __?__ 14 cm, 17 cm, and 31 cm long.

23. In $\square ABCD$, if $m\angle A > m\angle B$, then $\angle D$ is __?__ an acute angle.

24. Two obtuse triangles are __?__ similar.

25. Two lines perpendicular to a third line are __?__ perpendicular to each other.

Complete.

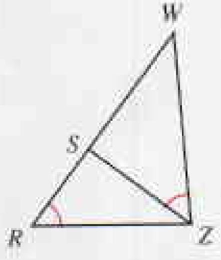
26. If $\frac{7}{x} = \frac{9}{10}$, then __?__ = __?__, and x = __?__.

B **27.** The sine of any acute angle must be greater than __?__ and less than __?__.

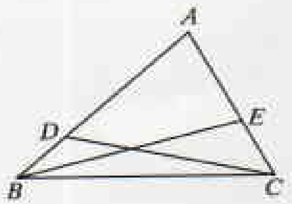
28. a. $\triangle RWZ \sim$ __?__

b. $\frac{RW}{?} = \frac{ZR}{?} = \frac{WZ}{?}$

c. $RW = 15$, $ZR = 10$, and $SZ = 8$.
$WZ =$ __?__ and $RS =$ __?__



29. Given: $AB > AC$; $\overline{BD} \cong \overline{EC}$
Prove: $BE > CD$



30. Given: $\frac{PR}{TR} = \frac{SR}{QR}$
Prove: $\angle S \cong \angle Q$